Topping Topoi

An introduction to sheaf and Grothendieck topos

Yichen Liu, Yuan Yao, Gene Tsai

Topology in and via Logic 2026

Table of contents

Introduction to Sheaf

How do we study structures of spaces?

Given a topological space $(X, \mathcal{T})$. We assign to each open set U a set of elements s which satisfy certain property ϕ within U:

$$U \longmapsto F(U) = \{s \mid U \vDash \phi(s)\}$$

Examples:

- $F(U) = C^0(U, \mathbb{R})$ (continuous real-valued functions on U)
- $F(U) = B(U, \mathbb{R})$ (bounded real-valued functions on U)
- $F(U) = K(U, \mathbb{R})$ (constant real-valued functions on U)

Section of F over U

$$s \in F(U)$$

What do we want the assignment to conform to?

For any open sets $U, V \in \mathcal{T}$ such that $U \subseteq V$, there is a restriction map $\rho_U^V : F(V) \to F(U)$ such that

1. for all open set U, $\rho_U^U = id_{F(U)}$
2. for all open sets $U \subseteq V \subseteq W$, $\rho_U^V \circ \rho_V^W = \rho_U^W$

The restriction maps for the previous examples are simply restricting the domain of functions.

Notation: for open sets $U \subseteq V$ and $s \in F(V)$, $s|_U := \rho_U^V(s)$

Presheaf on a topological space

A pair $(F, (\rho_U^V)_{U \subseteq V})$ where F is a map from $\mathcal{T}$ to **Sets** and $(\rho_U^V : F(V) \to F(U))$ is a collection of maps satisfying the above conditions

Relation between a section and its restrictions

Given a presheaf (F, ρ) and open set U. It would be nice that we can reconstruct a section of U by assembling its restrictions on an open cover of U. This property could be formalised as

Sheaf condition

Given an open set U and an open cover $(U_i)_{i \in I}$ of it.

1. For $s, t \in F(U)$, if $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$.
2. For any family of sections $(s_i \in F(U_i))_{i \in I}$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j \in I$, there is an $s \in F(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

Sheaf of a topolocial space

A sheaf is a presheaf satisfying sheaf property.

Rephrasing definitions in the language of category theory

A topology $\mathcal{T}$ on X is a poset ordered by $\subseteq$, which can be seen as a category where there is at most one morphism between any two objects.

Presheaf on $(X, \mathcal{T})$

A functor $F \in [\mathcal{T}^{op}, \mathbf{Sets}]$

Sheaf on $(X, \mathcal{T})$

A presheaf $F \in [\mathcal{T}^{op}, \mathbf{Sets}]$ such that for each open set U and open cover $(U_i)_{i \in I}$ of U, the diagram

$$F(U) \xrightarrow{\langle F(U \hookrightarrow U_i) \rangle_{i \in I}} \prod_{i \in I} F(U_i) \underset{F(U_i \cap U_j \hookrightarrow U_j) \circ \pi_j}{\overset{F(U_i \cap U_j \hookrightarrow U_i) \circ \pi_i}{\rightrightarrows}} \prod_{(i,j) \in I \times I} F(U_i \cap U_j)$$

forms an equalizer.

Generalizing Open Covers - Sieves

- Recall that in topological space $(X, \mathcal{T})$, the collection of all open subsets of U is all the inclusion maps from some object to U.
- In fact, this forms a presheaf $\mathrm{Hom}_{\mathcal{T}}(-, U)$.
- So categorically, we think of a collection of open subsets of U as a *"subpresheaf"* of $\mathrm{Hom}_{\mathcal{T}}(-, U)$.
- In order for these collections to be a presheaf, they must be downward closed, and they are so-called *"sieves."*

Sieve

A sieve S on an object C of category $\mathcal{C}$ is a subpresheaf of $\mathrm{Hom}_{\mathcal{C}}(-, C)$. That is, $S \subseteq \{f \mid \mathrm{cod}(f) = C\}$ such that

$$f \in S \Longrightarrow f \circ g \in S$$

for all g composable with f.

Generalizing Open Covers - Pullback Sieve

- Given an open set U, a collection of its subsets $\{U_i\}_{i\in I}$, and an open subset $V \subseteq U$.
- If the collection is a sieve on U, then the collection of intersections $\{U_i \cap V\}_{i\in I}$ is a subcollection of $\{U_i\}_{i\in I}$ thats forms a sieve on V.
- So we can generalize the notion of intersection with a subset as follows.

Pullback of Sieve

Given a sieve S on C and a morphism $f : D \to C$. The pullback of S along f is a sieve on D given by

$$f^*(S) := \{g : E \to D \mid f \circ g \in S\}.$$

Generalizing Open Covers - Grothendieck Topology

Grothendieck topology [1]

A Grothendieck topology Cov on category $\mathcal{C}$ assigns a sieve $\mathrm{Cov}(C)$ to each object C in $\mathcal{C}$, satisfying the follows:

1. (Stability): For each $f : D \to C$ in $\mathcal{C}$ and $S \in \mathrm{Cov}(C)$, $f^*(S) \in \mathrm{Cov}(D)$.
2. (Local Characterization): For any $S \in \mathrm{Cov}(C)$ and any sieve R on C, if $f^*(R) \in \mathrm{Cov}(D)$ for all $f : D \to C$ in S, then $R \in \mathrm{Cov}(C)$.
3. (Maximality): For each object C in $\mathcal{C}$, $\max(C) := \{f \mid \mathrm{cod}(f) = C\} \in \mathrm{Cov}(C)$.

It can be verivied that the usual notion of open cover in topological space $(X, \mathcal{T})$ gives a Grothendieck topology: if $S = \{U_i \hookrightarrow U\}_{i \in I}$ is a sieve on U, then

$$S \in \mathrm{Cov}(U) \iff \bigcup_{i \in I} U_i = U.$$

An example of a presheaf that fails sheaf condition

Example. (bounded functions)

- Consider $(\mathbb{R}, \mathcal{T}_{Euc})$ and $B \in [\mathcal{T}_{Euc}^{op}, \mathbf{Sets}]$ (the presheaf of bounded functions on $\mathbb{R}$.)
- Let $(U_i = (i-1, i+1))_{i\in\mathbb{Z}}$ which is an open cover of $\mathbb{R}$.
- For each $i \in \mathbb{Z}$, define $f_i : U_i \to \mathbb{R}$ by $f_i(x) = x$, then it is obvious that $f_i \in B(U_i)$.
- Moreover, we have $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ for all $i, j \in \mathbb{Z}$.
- However, the only function f on $\mathbb{R}$ such $f|_{U_i} = f_i$ for all i is the identity function, which is not in $B(\mathbb{R})$.

An example of a presheaf that fails sheaf condition

The example shows that not all local property can be *"glued"* together to form a global property.

Question

Is there a canonical way to turn a presheaf into sheaf?

Sheafification !

Sheafifiying Presheaves

Stalks

Given a presheaf $F \in \mathrm{PSh}(\mathcal{T})$

Stalk

The stalk of F at x is the filtered colimit

$$F_x := \operatorname*{colim}_{U \ni x} F(U) \quad .$$

(Note that $\{U \in \mathcal{T} \mid x \in U\}$ is indeed a prime filter.)

Remark.

$$F_x \cong \left(\bigsqcup_{U \ni x} F(U) \right) \Big/ \sim_x$$

where $s \in F(U) \sim_x t \in F(V)$ iff there is some open set $W \subseteq U \cap V$ such that $x \in W$ and $s|_W = t|_W$.

Germs

Germ

Given a presheaf F, an open set U, and a point $x \in U$. There is a mapping

$$F(U) \longrightarrow F_x$$
$$s \longmapsto s_x$$

sending a section to an equivalence class of sections over $\sim_x$. Such s_x is called the germ of s at x.

Properties about germs and stalks

Let $(X, \mathcal{T})$ be a topological space, and $F \in \mathrm{Sh}(\mathcal{T})$.

Lemma

1. The map

$$F(U) \to \prod_{x \in U} F_x$$
$$s \mapsto (s_x)_{x \in U}$$

   is injective.
2. Suppose $U \in \mathcal{T}$ and $s, t \in F(U)$. If $s_x = t_x \in F_x$ for all $x \in U$, then $s = t \in F(U)$.

[3]

Properties about germs and stalks

Let $(X, \mathcal{T})$ be a topological space, $F, G \in \mathrm{Sh}(T)$, and $\varphi : F \to G$ be a morphism of sheaves (i.e. a natural transformation.)

Proposition

If the induced map

$$\begin{aligned} \varphi_x : F_x &\longrightarrow G_x \\ s_x = [(U, s)] &\longmapsto [(U, \varphi_U(s))] = \varphi_U(s)_x \end{aligned}$$

is an bijection for all $x \in X$, then φ is an isomorphism. [3]

Sheafification

Let $(X, \mathcal{T})$ be a topological space.

Theorem (Sheafification)

The inclusion $\iota : \mathrm{Sh}(\mathcal{T}) \hookrightarrow \mathrm{PSh}(\mathcal{T})$ admits a left adjoint $L : \mathrm{PSh}(\mathcal{T}) \to \mathrm{Sh}(\mathcal{T})$ called the *sheafification*.

Universal Property of Sheafification

Given a presheaf F, a sheaf G, and a morphism $\sigma : F \to \iota(G)$. There is a unique $\eta : L(F) \to G$ such that the diagram

$$\begin{array}{ccc} F & \longrightarrow & \iota(L(F)) \\ & \overset{\sigma}{\searrow} & \big\downarrow \exists! \eta \\ & & \iota(G) \end{array}$$

commutes.

This can be understood as that $L(F)$ is a certain kind of *”best approximation of F”* in the sheaf category.

Constructing Sheafification

Construction of L

Given a presheaf $F \in \mathrm{PSh}(\mathcal{T})$.

$$L(F)(U) := \left\{ (s_x)_{x \in U} \in \prod_{x \in U} F_x \;\middle|\; \begin{array}{l} \text{for all } x \in U, \\ \text{there is some } U_x \subseteq U \text{ containing } x \\ \text{such that there is some } s' \in F(U_x) \\ \text{such that } s_y = s'_y \text{ for all } y \in U_x \end{array} \right\}$$

- By taking the elements of stalks, we ensure that the sheafification satisfies the locality condition of sheaf.
- By requiring the RHS condition, we make these elements *"glueable"*.

From Sheaves to Topoi

Historical motivation of generalizing Sheaves to Topoi

- In mid-20th century, Weil conjecture is a research focus of algebraic geometry.

Historical motivation of generalizing Sheaves to Topoi

- In mid-20th century, Weil conjecture is a research focus of algebraic geometry.
- A key step in proof of the conjecture was to construct étale cohomology.

Historical motivation of generalizing Sheaves to Topoi

- In mid-20th century, Weil conjecture is a research focus of algebraic geometry.
- A key step in proof of the conjecture was to construct étale cohomology.
- Zariski topology is a convenient topology to assigned on algebraic varieties, but it is too coarse.

Historical motivation of generalizing Sheaves to Topoi

- In mid-20th century, Weil conjecture is a research focus of algebraic geometry.
- A key step in proof of the conjecture was to construct étale cohomology.
- Zariski topology is a convenient topology to assigned on algebraic varieties, but it is too coarse.
- Grothendieck came up with *Grothendieck topos*, which is a category of sheaves involving *Grothendieck topology*.

Historical motivation of generalizing Sheaves to Topoi

- In mid-20th century, Weil conjecture is a research focus of algebraic geometry.
- A key step in proof of the conjecture was to construct étale cohomology.
- Zariski topology is a convenient topology to assigned on algebraic varieties, but it is too coarse.
- Grothendieck came up with *Grothendieck topos*, which is a category of sheaves involving *Grothendieck topology*.
- These tools enables the construction of the desired cohomology.

Recall: Grothendieck topology

Grothendieck topology

A Grothendieck topology Cov on category $\mathcal{C}$ assigns a sieve $\mathrm{Cov}(C)$ to each object C in $\mathcal{C}$, satisfying the follows:

1. (Stability): For each $f : D \to C$ in $\mathcal{C}$ and $S \in \mathrm{Cov}(C)$, $f^*(S) \in \mathrm{Cov}(D)$.
2. (Local Characterization): For any $S \in \mathrm{Cov}(C)$ and any sieve R on C, if $f^*(R) \in \mathrm{Cov}(D)$ for all $f : D \to C$ in S, then $R \in \mathrm{Cov}(C)$.
3. (Maximality): For each object C in $\mathcal{C}$, $\mathsf{max}(C) := \{f \mid \mathrm{cod}(f) = C\} \in \mathrm{Cov}(C)$.

Site

A pair $(\mathcal{C}, \mathrm{Cov})$ where $\mathcal{C}$ is a category and Cov is a Grothendieck topology on $\mathcal{C}$.
A site is *small* if $\mathcal{C}$ is small.

Sheaves on a site [2]

Given a presheaf $F \in [\mathcal{C}^{op}, \mathsf{Sets}]$.

Matching family

Given a sieve S on object C in $\mathcal{C}$. A matching family assigns to each $f : D \to C$ in S an $x_f \in F(D)$ such that

$$F(g)(x_f) = x_{f \circ g} \quad \forall g : E \to D.$$

Amalgamation

An amalgamation for the above matching family is an element $x \in F(C)$ such that $F(f)(x) = x_f$ for all $f \in S$.

Sheaf on a site

F is a sheaf on site (C, Cov) iff for all object $C \in \mathcal{C}$ and $S \in \mathrm{Cov}(C)$, every matching family of S has a unique amalgamation.

Reformulating Sheaves via Limits

- Let S be a covering sieve on C.
- Notice that the set of "matching families" for S is precisely the limit of $\mathcal{F}$ restricted to the sieve.
- The "amalgamation" condition says the map from $\mathcal{F}(C)$ to this limit is a bijection.

Limit Definition of a Sheaf

A presheaf $\mathcal{F}$ is a sheaf if and only if for every object C and every covering sieve $S \in \mathrm{Cov}(C)$, the canonical map is a bijection:

$$\mathcal{F}(C) \xrightarrow{\cong} \lim_{D \xrightarrow{f} C \in S} \mathcal{F}(D)$$

This perspective allows us to "force" the sheaf condition using limits and colimits.

The Plus Construction ($\mathcal{F}^\dagger$)

To turn a presheaf into a sheaf, we construct a new presheaf $\mathcal{F}^\dagger$ that "fixes" the failure of the limit condition.

Construction

For any presheaf $\mathcal{F}$ and object C, define:

$$\mathcal{F}^\dagger(C) := \operatorname*{colim}_{S \in \mathrm{Cov}(C)} \left(\lim_{D \xrightarrow{f} C \in S} \mathcal{F}(D) \right)$$

where the colimit is taken over the poset of covering sieves (ordered by reverse inclusion).

- **Inner Limit:** Constructs "potential sections" (matching families) for a specific cover S.
- **Outer Colimit:** Identifies matching families that agree on a finer cover (germs of sections).

Step 1: Separated Presheaves

The first application of the plus construction ensures uniqueness of amalgamations, but not necessarily existence.

Separated Presheaf

A presheaf $\mathcal{F}$ is **separated** if the map $\mathcal{F}(C) \to \prod_{f \in S} \mathcal{F}(\mathrm{dom}(f))$ is injective for every cover S. (Equivalently, matching families have *at most one* amalgamation).

First Application

For *any* presheaf $\mathcal{F}$, the presheaf $\mathcal{F}^\dagger$ is separated.

The plus construction removes "ghost elements" that vanish locally.

Step 2: The Sheafification Theorem

Second Application

If $\mathcal{F}$ is already a **separated** presheaf, then $\mathcal{F}^\dagger$ is a **sheaf**.

- Therefore, applying the construction twice yields the sheafification.

Theorem (Sheafification Theorem)

The inclusion functor $\mathrm{Sh}(\mathcal{C}) \hookrightarrow Fun(\mathcal{C}^{op}, Set)$ has a left adjoint L, called ***sheafification****. It is given by:*

$$L(\mathcal{F}) := (\mathcal{F}^\dagger)^\dagger$$

Furthermore, L preserves finite limits (is left exact).

Grothendieck topos

The collection of sheaves on $(\mathcal{C}, \mathrm{Cov})$ together with natural transformations between them forms a category $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$.
A Grothendieck topos is a category that is equivalent to $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ for some small site $(\mathcal{C}, \mathrm{Cov})$.

Giraud's Theorem: Characterization of Topoi

Theorem (Giraud)

Let $\mathcal{X}$ be a category. The following conditions are equivalent:

1. *$\mathcal{X}$ is a Grothendieck topos (i.e., $\mathcal{X} \simeq \mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ for a small site $(\mathcal{C}, \mathrm{Cov})$).*
2. *There exists a small category $\mathcal{C}$ and a fully faithful embedding $\mathcal{X} \hookrightarrow \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ admitting a left adjoint L that preserves finite limits.*
3. *$\mathcal{X}$ satisfies* ***Giraud's Axioms****:*
   - **(G1)** *$\mathcal{X}$ admits finite limits.*
   - **(G2)** *Every equivalence relation in $\mathcal{X}$ is effective.*
   - **(G3)** *$\mathcal{X}$ has disjoint small coproducts.*
   - **(G4)** *Effective epimorphisms are stable under pullback.*
   - **(G5)** *Coproducts differ commute with pullback (universality).*
   - **(G6)** *$\mathcal{X}$ has a set of generators.*

[4]

Characterization (2): Topos as a Localization

- We can interpret the left adjoint $L : \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set}) \to \mathcal{X}$ as a **localization functor**.
- The sheafification process essentially "forces" certain morphisms to become isomorphisms.

Inverting Local Isomorphisms

Let Σ be the collection of **local isomorphisms** (morphisms $\alpha : \mathcal{F} \to \mathcal{G}$ that become isomorphisms locally on a cover).

- The functor L maps every $\alpha \in \Sigma$ to an isomorphism in $\mathcal{X}$.
- Conversely, any functor inverting Σ factors uniquely through L.

- Thus, we view the topos $\mathcal{X}$ as the category obtained from presheaves by **formally inverting** all locally equivalent morphisms.
- **Summary:** A Grothendieck topos is a *left exact localization* of a presheaf category.

Generalization to ∞-Topoi

- The characterization of a topos as a **left exact localization** (Condition 2) is the most robust definition for generalization.
- It allows us to pass from "sets" to "spaces" (homotopy types) seamlessly.

The ∞-Categorical Analogy

- **1-Topos:** A category $\mathcal{X}$ is a Grothendieck topos if it is a left exact localization of a presheaf category $\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$.
- **∞-Topos:** An ∞-category $\mathcal{X}$ is an ∞-topos if it is a left exact localization of an ∞-category of presheaves $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$, where $\mathcal{S}$ is the ∞-category of spaces (animas).

- In this framework, "sheafification" becomes a localization functor L that enforces **homotopical descent**.
- This definition avoids the immediate complexity of "sites with homotopy coherent covers," making the theory much cleaner to set up.

∞-Topoi: Descent for Sheaves of Spaces

Let $\mathcal{X}$ be a small ∞-category equipped with a Grothendieck topology, and $F : \mathcal{X}^{op} \to \mathcal{S}$ be a presheaf of spaces.

The Cech Cosimplicial Object: Given a cover $\mathcal{U} = \{U_i \to X\}$ in $\mathcal{X}$, the evaluation of F on the Cech nerve $\check{C}(\mathcal{U})$ yields a cosimplicial diagram in $\mathcal{S}$:

$$F(X) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_{ij}) \overset{\to}{\underset{\to}{\to}} \dots$$

The Descent Condition (Sheaf Property)

The presheaf F is a **sheaf** if for every cover $\mathcal{U}$, the map to the (homotopy) limit is a (weak) homotopy equivalence of spaces:

$$F(X) \xrightarrow{\sim} \lim_{\Delta} F(\check{C}(\mathcal{U}))$$

(Here, the limit is taken in the ∞-category $\mathcal{S}$)

Topos as a Semantics for Logic

- Giraud's theorem tells us exactly what categorical structures exist in every topos.
- Remarkably, these structures correspond one-to-one with the operations of first-order intuitionistic logic.
- This allows us to view a topos not just as a "generalized space," but as a "mathematical universe" where logic can be performed.

Categorical Structure (Giraud)	Logical Operation
Finite Limits (G1)	Conjunction ($\wedge$), Truth ($\top$), Substitution
Coproducts (G3)	Disjunction ($\vee$), Falsehood ($\perp$)
Subobject Lattices	Propositional Logic
Adjoints to Pullback	Quantifiers ($\exists, \forall$)

Interpreting IPL in topos

Fix propositional variables $\mathsf{Var} = \{p, q, \ldots\}$. Formulas:

$$\varphi ::= p \mid \top \mid \bot \mid (\varphi \wedge \psi) \mid (\varphi \vee \psi) \mid (\varphi \to \psi), \qquad \neg\varphi := (\varphi \to \bot).$$

1. What should be the "truth object" of propositions?
2. How should the "truth object" of propositions interact with connectives?

Valuation

1. What should be the “truth object” of propositions?

$$[\![\varphi]\!] \in \mathrm{Sub}(1) \cong \mathcal{E}(1, \Omega),$$

$\mathrm{Sub}(1)$: poset of subobjects of 1, with $A \leq B$ if $A \hookrightarrow B$

In $\mathcal{E} = \mathrm{PSh}(\mathcal{C})$:

- 1 is the stage-wise terminal object, i.e. sending each $C \in \mathcal{C}$ to singleton $\{*\}$ (with unique restriction).
- A subobject of 1 (a subterminal) is a subpresheaf P of 1 with each section either $\{*\}$ or $\emptyset$ stable under restriction, meaning:

$$P(C) = \{*\} \text{ and } \alpha : D \to C \implies P(D) = \{*\}.$$

Valuation

2. How should the "truth object" of propositions interact with connectives? –What operations do we have on $\mathrm{Sub}(1)$?

Proposition

In any topos (in particular in $\mathrm{PSh}(\mathcal{C})$ or $\mathrm{Sh}(\mathcal{C})$), the poset $\mathrm{Sub}(1)$ forms a Heyting algebra. Concretely, for subobjects $A \hookrightarrow 1$ and $B \hookrightarrow 1$:

- $\top := 1 \hookrightarrow 1, \qquad \bot := 0 \hookrightarrow 1.$
- $A \sqcap B := A \times B \hookrightarrow 1$ (pullback / intersection).
- $A \sqcup B := \mathrm{im}\big(A \amalg B \to 1\big) \hookrightarrow 1$ (join / union).
- $A \Rightarrow B$ is the *largest* subobject $C \hookrightarrow 1$ such that $C \sqcap A \leq B$, i.e. for all $D \hookrightarrow 1$,

$$D \leq (A \Rightarrow B) \iff (D \sqcap A) \leq B.$$

Valuation

Valuation

A valuation is a function

$$v : \mathsf{Var} \to \mathrm{Sub}(1) :: p \mapsto v(p) =: [\![p]\!] \hookrightarrow 1.$$

Valuation

Valuation

A valuation is a function

$$v : \mathsf{Var} \to \mathrm{Sub}(1) :: p \mapsto v(p) =: [\![p]\!] \hookrightarrow 1.$$

Then the valuation v can be extended to all formulas, i.e. $[\![-]\!] : \mathrm{Form}_{\mathsf{IPL}} \to \mathrm{Sub}(1)$, via the algebraic operations on $\mathrm{Sub}(1)$ inductively. For example:

$$[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \sqcap [\![\psi]\!], \quad [\![\varphi \vee \psi]\!] = [\![\varphi]\!] \sqcup [\![\psi]\!],$$

Presheaf semantics: stages and forcing

Consider a presheaf topos $\mathcal{E} = \mathrm{PSh}(\mathcal{C}) = \mathbf{Sets}^{\mathcal{C}^{op}}$.

Presheaf semantics: stages and forcing

Consider a presheaf topos $\mathcal{E} = \mathrm{PSh}(\mathcal{C}) = \mathbf{Sets}^{\mathcal{C}^{op}}$.

Forcing at an object $C \in \mathcal{C}$

For a subterminal $P \hookrightarrow 1$, define

$$C \Vdash P \quad :\Longleftrightarrow \quad P(C) = \{*\}$$

Since subterminals are stable under restriction, presheaf forcing is *persistent*: if $C \Vdash P$ and $\alpha : D \to C$ then $D \Vdash P$.

Presheaf semantics: stages and forcing

Consider a presheaf topos $\mathcal{E} = \mathrm{PSh}(\mathcal{C}) = \mathbf{Sets}^{\mathcal{C}^{op}}$.

Forcing at an object $C \in \mathcal{C}$

For a subterminal $P \hookrightarrow 1$, define

$$C \Vdash P \quad :\Longleftrightarrow \quad P(C) = \{*\}$$

Now each proposition φ is interpreted to a *subterminal* $[\![\varphi]\!] \hookrightarrow 1$.
Inductively, for φ, ψ: $C \Vdash [\![\varphi \vee \psi]\!] \iff ?$

Presheaf semantics: stages and forcing

Consider a presheaf topos $\mathcal{E} = \mathrm{PSh}(\mathcal{C}) = \textbf{Sets}^{\mathcal{C}^{op}}$.

Forcing at an object $C \in \mathcal{C}$

For a subterminal $P \hookrightarrow 1$, define

$$C \Vdash P \quad :\Longleftrightarrow \quad P(C) = \{*\}$$

Now each proposition φ is interpreted to a *subterminal* $[\![\varphi]\!] \hookrightarrow 1$.
Inductively, for φ, ψ: $C \Vdash [\![\varphi \vee \psi]\!] \iff C \Vdash [\![\varphi]\!] \sqcup [\![\psi]\!]$, where $[\![\varphi]\!] \sqcup [\![\psi]\!]$ is the join of $[\![\varphi]\!]$ and $[\![\psi]\!]$ in Heyting algebra $\mathrm{Sub}(1)$.

Presheaf semantics: stages and forcing

Consider a presheaf topos $\mathcal{E} = \mathrm{PSh}(\mathcal{C}) = \textbf{Sets}^{\mathcal{C}^{op}}$.

Forcing at an object $C \in \mathcal{C}$

For a subterminal $P \hookrightarrow 1$, define

$$C \Vdash P \quad :\Longleftrightarrow \quad P(C) = \{*\}$$

Now each proposition φ is interpreted to a *subterminal* $[\![\varphi]\!] \hookrightarrow 1$.
Inductively, for φ, ψ: $C \Vdash [\![\varphi \vee \psi]\!] \iff C \Vdash [\![\varphi]\!] \sqcup [\![\psi]\!]$, where $[\![\varphi]\!] \sqcup [\![\psi]\!]$ is the join of $[\![\varphi]\!]$ and $[\![\psi]\!]$ in Heyting algebra $\mathrm{Sub}(1)$.

$$\begin{aligned}
(A \sqcup B)(C) &\cong \mathrm{im}\big(A(C) \amalg B(C) \to 1(C)\big) \subseteq \{*\} \\
(A \sqcup B)(C) = \{*\} &\iff A(C) \amalg B(C) \neq \varnothing \\
&\iff A(C) = \{*\} \text{ or } B(C) = \{*\} \\
&\iff C \Vdash A \text{ or } C \Vdash B
\end{aligned}$$

Presheaf semantics: stages and forcing

Consider a presheaf topos $\mathcal{E} = \mathrm{PSh}(\mathcal{C}) = \mathbf{Sets}^{\mathcal{C}^{op}}$.

Forcing at an object $C \in \mathcal{C}$

For a subterminal $P \hookrightarrow 1$, define

$$C \Vdash P \quad :\Longleftrightarrow \quad P(C) = \{*\}$$

Now each proposition φ is interpreted to a *subterminal* $[\![\varphi]\!] \hookrightarrow 1$.
Inductively, for φ, ψ:

$$C \Vdash [\![\varphi \to \psi]\!] \iff C \Vdash [\![\varphi]\!] \Rightarrow [\![\psi]\!]$$

for Heyting implication in $\mathrm{Sub}(1)$.

$$\begin{aligned}(A \Rightarrow B)(C) = \{*\} &\iff \forall f : D \to C,\ \big(A(D) = \{*\} \Rightarrow B(D) = \{*\}\big)\\ &\iff \forall f : D \to C,\ \big(D \Vdash A \Rightarrow D \Vdash B\big).\end{aligned}$$

Presheaf semantics: forcing clauses

For any presheaf topos $\mathcal{E} = \mathrm{PSh}(\mathcal{C})$, $C \in \mathcal{C}$, valuation v and formula φ, define $\mathcal{E}, C, v \models \varphi$ iff $C \Vdash [\![\varphi]\!]^v$, we get the following semantics:

Presheaf semantics

$$\mathcal{E}, C, v \models \top \text{ always}, \quad \mathcal{E}, C, v \models \bot \text{ never}$$

$$\mathcal{E}, C, v \models p \iff C \Vdash v(p) = [\![p]\!]^v, \quad p \in \mathrm{Var}$$

Inductively:

$\mathcal{E}, C, v \models (\varphi \wedge \psi)$ iff ($\mathcal{E}, C, v \models \varphi$ and $\mathcal{E}, C, v \models \psi$).

$\mathcal{E}, C, v \models (\varphi \vee \psi)$ iff ($\mathcal{E}, C, v \models \varphi$ or $\mathcal{E}, C, v \models \psi$).

$\mathcal{E}, C, v \models (\varphi \to \psi)$ iff $\forall \alpha : D \to C$, $\big(\mathcal{E}, D, v \models \varphi$ implies $\mathcal{E}, D, v \models \psi\big)$.

And negation is derived as:

$$C, v \models \neg\varphi \text{ iff } \forall \alpha : D \to C,\ D, v \not\models \varphi$$

Validity

For a presheaf topos $\mathcal{E} = \mathrm{PSh}(\mathcal{C}) = \mathbf{Sets}^{\mathcal{C}^{op}}$: a formula φ is valid over $\mathcal{E}$ (denoted by $\mathcal{E} \models \varphi$) if for all valuations v, and for all $C \in \mathcal{C}$, $C, v \models \varphi$.

For a class $\mathscr{E}$ of presheaf toposes: φ is valid over $\mathscr{E}$ if for all $\mathcal{E} \in \mathscr{E}, \mathcal{E} \models \varphi$.

Denote the collection of valid formulas on a topos by

$$\mathsf{Log}(\mathcal{E}) = \{\varphi \mid \mathcal{E} \models \varphi\}$$

and similarly

$$\mathsf{Log}(\mathscr{E}) = \{\varphi \mid \forall \mathcal{E} \in \mathscr{E}, \mathcal{E} \models \varphi\}$$

Validity

For a presheaf topos $\mathcal{E} = \mathrm{PSh}(\mathcal{C}) = \mathbf{Sets}^{\mathcal{C}^{op}}$: a formula φ is valid over $\mathcal{E}$ (denoted by $\mathcal{E} \models \varphi$) if for all valuations v, and for all $C \in \mathcal{C}$, $C, v \models \varphi$.

For a class $\mathscr{E}$ of presheaf toposes: φ is valid over $\mathscr{E}$ if for all $\mathcal{E} \in \mathscr{E}, \mathcal{E} \models \varphi$.

Denote the collection of valid formulas on a topos by

$$\mathsf{Log}(\mathcal{E}) = \{\varphi \mid \mathcal{E} \models \varphi\}$$

and similarly

$$\mathsf{Log}(\mathscr{E}) = \{\varphi \mid \forall \mathcal{E} \in \mathscr{E}, \mathcal{E} \models \varphi\}$$

Soundness

IPC $\subseteq \mathsf{Log}(\mathrm{PSh}(\mathcal{C}))$ for any small category $\mathcal{C}$.

Example: Kripke semantics for IPC

Let $\mathcal{F} := (W, R)$ be a Kripke frame for **IPC**, i.e. R is a partial order on W. Think of $\mathcal{F}$ as a category with a unique arrow $u \to w$ iff wRu. Consider $\mathcal{E} = \mathrm{PSh}(\mathcal{F}) = \mathbf{Sets}^{\mathcal{F}^{op}}$.

Example: Kripke semantics for IPC

Let $\mathcal{F} := (W, R)$ be a Kripke frame for **IPC**, i.e. R is a partial order on W. Think of $\mathcal{F}$ as a category with a unique arrow $u \to w$ iff wRu. Consider $\mathcal{E} = \mathrm{PSh}(\mathcal{F}) = \mathbf{Sets}^{\mathcal{F}^{op}}$.

Valuations as persistent sets of worlds

The map $P \hookrightarrow 1 \mapsto \{w \mid P(w) = \{*\}\}$ is a bijection between $\mathrm{Sub}_{\mathrm{PSh}(\mathcal{F})}(1)$ and R-persistent subsets of W; under this bijection, $[\![\varphi]\!]_v$ corresponds to the Kripke truth set of φ for the induced valuation V.

Example: Kripke semantics for IPC

Let $\mathcal{F} := (W, R)$ be a Kripke frame for **IPC**, i.e. R is a partial order on W. Think of $\mathcal{F}$ as a category with a unique arrow $u \to w$ iff wRu. Consider $\mathcal{E} = \mathrm{PSh}(\mathcal{F}) = \mathbf{Sets}^{\mathcal{F}^{op}}$.

Valuations as persistent sets of worlds

The map $P \hookrightarrow 1 \mapsto \{w \mid P(w) = \{*\}\}$ is a bijection between $\mathrm{Sub}_{\mathrm{PSh}(\mathcal{F})}(1)$ and R-persistent subsets of W; under this bijection, $[\![\varphi]\!]_v$ corresponds to the Kripke truth set of φ for the induced valuation V.

A valuation v in $\mathcal{E}$ assigns each variable p a subobject $v(p) \hookrightarrow 1$. Stability under restriction implies the persistence of presheaf forcing, i.e.

$$v(p)(w) = \{*\} \text{ and } u \to w \implies v(p)(u) = \{*\}$$

Therefore, $V(p) := \{\, w \in W \mid v(p)(w) = \{*\}\,\}$ is a *R-upset* (a valuation in a Kripke model for **IPC**).

Example: Kripke semantics for IPC

Proposition

Let $\mathcal{F} = (W, R)$ be a Kripke frame and $\mathcal{E} = \mathrm{PSh}(\mathcal{F})$. Given a presheaf valuation $v : \mathsf{Var} \to \mathrm{Sub}_{\mathcal{E}}(1)$, let V be the induced Kripke valuation

$$V(p) := \{\, w \in W \mid v(p)(w) = \{*\} \,\}.$$

Then for every formula $\varphi \in \mathsf{IPL}$,

$$V(\varphi) \;=\; \{\, w \in W \mid [\![\varphi]\!]_v(w) = \{*\} \,\},$$

i.e. $\mathcal{F}, V, w \models \varphi$ iff $[\![\varphi]\!]_v(w) = \{*\}$ iff $\mathcal{E}, w, v \models_{\mathrm{PSh}} \varphi$.

Example: Kripke semantics for IPC

Proposition

Let $\mathcal{F} = (W, R)$ be a Kripke frame and $\mathcal{E} = \mathrm{PSh}(\mathcal{F})$. Given a presheaf valuation $v : \mathsf{Var} \to \mathrm{Sub}_{\mathcal{E}}(1)$, let V be the induced Kripke valuation

$$V(p) := \{\, w \in W \mid v(p)(w) = \{*\} \,\}.$$

Then for every formula $\varphi \in \mathsf{IPL}$,

$$V(\varphi) \;=\; \{\, w \in W \mid [\![\varphi]\!]_v(w) = \{*\} \,\},$$

i.e. $\mathcal{F}, V, w \models \varphi$ iff $[\![\varphi]\!]_v(w) = \{*\}$ iff $\mathcal{E}, w, v \models_{\mathrm{PSh}} \varphi$.

So presheaf semantics on poset presheaves is just Kripke semantics for intuitinistic logic.

Example: Kripke semantics for IPC

Proposition

Let $\mathcal{F} = (W, R)$ be a Kripke frame and $\mathcal{E} = \mathrm{PSh}(\mathcal{F})$. Given a presheaf valuation $v : \mathsf{Var} \to \mathrm{Sub}_{\mathcal{E}}(1)$, let V be the induced Kripke valuation

$$V(p) := \{\, w \in W \mid v(p)(w) = \{*\} \,\}.$$

Then for every formula $\varphi \in \mathsf{IPL}$,

$$V(\varphi) \;=\; \{\, w \in W \mid [\![\varphi]\!]_v(w) = \{*\} \,\},$$

i.e. $\mathcal{F}, V, w \models \varphi$ iff $[\![\varphi]\!]_v(w) = \{*\}$ iff $\mathcal{E}, w, v \models_{\mathrm{PSh}} \varphi$.

Corollary

$\mathbf{IPC} = \mathsf{Log}\Big(\{\mathrm{PSh}(\mathcal{P}) \mid \mathcal{P} \textit{ is a poset}\}\Big) = \mathsf{Log}\Big(\{\mathrm{PSh}(\mathcal{C}) \mid \mathcal{C} \textit{ is small}\}\Big)$

Sheaf semantics

For a sheaf topos $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$, idea is the same: evaluate variables into $\mathrm{Sub}(1)$, compute composite formulas via algebraic operations in $\mathrm{Sub}(1)$. But in sheaf topos:

Sheaf semantics

For a sheaf topos $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$, idea is the same: evaluate variables into $\mathrm{Sub}(1)$, compute composite formulas via algebraic operations in $\mathrm{Sub}(1)$. But in sheaf topos:

- 1 remains the same (all stages agree with each other)

Sheaf semantics

For a sheaf topos $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$, idea is the same: evaluate variables into $\mathrm{Sub}(1)$, compute composite formulas via algebraic operations in $\mathrm{Sub}(1)$. But in sheaf topos:

- 1 remains the same (all stages agree with each other)
- $P \in \mathrm{Sub}(1) : \mathcal{C} \to \{\emptyset, \{*\}\}$ satisfying:
  1. Stable under restriction
  2. Sheaf condition: for any covering sieve $S \in \mathrm{Cov}(C)$, if

$$\Big(\forall\, \alpha : D \to C \in S,\ P(D) = \{*\}\Big) \implies P(C) = \{*\}.$$

     Because for subterminals: there is at most one amalgamation for a matching family, i.e. $\{*\}$. So the amalgamation condition reduces to the existence of such an amagamation.

Sheaf semantics

For a sheaf topos $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$, idea is the same: evaluate variables into $\mathrm{Sub}(1)$, compute composite formulas via algebraic operations in $\mathrm{Sub}(1)$. But in sheaf topos:

- 1 remains the same (all stages agree with each other)
- $P \in \mathrm{Sub}(1) : \mathcal{C} \to \{\emptyset, \{*\}\}$ satisfying:
  1. Stable under restriction
  2. Sheaf condition: for any covering sieve $S \in \mathrm{Cov}(C)$, if

     $$\Big(\forall \alpha : D \to C \in S,\ P(D) = \{*\}\Big) \implies P(C) = \{*\}.$$

     Because for subterminals: there is at most one amalgamation for a matching family, i.e. $\{*\}$. So the amalgamation condition reduces to the existence of such an amagamation.
- Heyting operations on $\mathrm{Sub}(1)$?
  - Only $\vee$ changes

Sheaf semantics: $\vee$ becomes local

In $\mathrm{PSh}(\mathcal{C})$:

$$C, v \models (\varphi \vee \psi) \text{ iff } (C, v \models \varphi \text{ or } C, v \models \psi).$$

In $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$, local truth on covers implies global truth:

$$\boxed{\begin{aligned} C, v \models (\varphi \vee \psi) \text{ iff } \exists(\alpha_i : D_i \to C)_{i \in I} \in \mathrm{Cov}(C) \\ \text{s.t. } \forall i,\ (D_i, v \models \varphi \text{ or } D_i, v \models \psi) \end{aligned}}$$

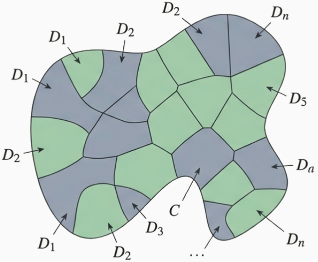


Image generated by NotebookLM

Sheaf semantics: $\vee$ becomes local

In $\mathrm{PSh}(\mathcal{C})$:

$$C, v \models (\varphi \vee \psi) \text{ iff } (C, v \models \varphi \text{ or } C, v \models \psi).$$

In $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$, local truth on covers implies global truth:

$$\boxed{\begin{aligned} C, v \models (\varphi \vee \psi) \text{ iff } \exists(\alpha_i : D_i \to C)_{i \in I} \in \mathrm{Cov}(C) \\ \text{s.t. } \forall i,\ (D_i, v \models \varphi \text{ or } D_i, v \models \psi) \end{aligned}}$$

In $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ the join in $\mathrm{Sub}(1)$ is computed as the image of a coproduct (involving colimits), but colimits in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ are obtained by sheafifying presheaf colimits. Therefore $\vee$ becomes local (witnessed only after passing to a cover).

Sheaf on a space

Recall: For a topological space $(X, \mathcal{T})$, $\mathrm{Sh}(\mathcal{T})$ denotes the sheaf on $(\mathcal{T}, \subseteq)$. A covering sieve in $\mathrm{Cov}(U)$ is exactly the $\subseteq$-downward closure of an open over of U.

Proposition

For any $\mathrm{Sh}(\mathcal{T})$, there is a bijection $\Theta : \mathrm{Sub}(1) \rightarrowtail\!\!\!\!\twoheadrightarrow \mathcal{T}$.

For each open $U \in \tau$, define the subterminal sheaf $1_U \hookrightarrow 1$ by

$$1_U(V) := \begin{cases} \{*\} & \text{if } V \subseteq U, \\ \varnothing & \text{otherwise.} \end{cases}$$

Conversely, for a subobject $A \hookrightarrow 1$ in $\mathrm{Sh}(X)$, let

$$U_A \ := \bigcup \{ V \in \mathcal{T} \mid A(V) = \{*\} \}.$$

$U \mapsto (1_U \hookrightarrow 1)$ and $(A \hookrightarrow 1) \mapsto U_A$ are mutually inverse.

Sheaf on a space

The poset $(\mathcal{T}, \subseteq)$ carries a Heyting structure $(\mathcal{T}, X, \emptyset, \cup, \cap, \Rightarrow)$.

Sheaf on a space

The poset $(\mathcal{T}, \subseteq)$ carries a Heyting structure $(\mathcal{T}, X, \emptyset, \cup, \cap, \Rightarrow)$.

The Heyting implication $U \Rightarrow V$ is given by the *largest* open set W such that $W \cap U \subseteq V$. Equivalently,

$$W \subseteq (U \Rightarrow V) \iff (W \cap U) \subseteq V,$$

so in particular

$$U \Rightarrow V \; = \; \mathrm{int}\big((X \setminus U) \cup V\big)$$

and negation is derived as

$$\neg U \; := \; (U \Rightarrow \varnothing) = \mathrm{int}(X \setminus U)$$

.

Sheaf on a space

The poset $(\mathcal{T}, \subseteq)$ carries a Heyting structure $(\mathcal{T}, X, \emptyset, \cup, \cap, \Rightarrow)$.

The Heyting implication $U \Rightarrow V$ is given by the *largest* open set W such that $W \cap U \subseteq V$. Equivalently,

$$W \subseteq (U \Rightarrow V) \iff (W \cap U) \subseteq V,$$

so in particular

$$U \Rightarrow V \;=\; \mathrm{int}\big((X \setminus U) \cup V\big)$$

and negation is derived as

$$\neg U \;:=\; (U \Rightarrow \varnothing) = \mathrm{int}(X \setminus U)$$

.

There's more!

Θ is a Heyting isomorphism between $(\mathrm{Sub}_{\mathrm{Sh}(\mathcal{T})}(1), \top, \bot, \sqcup, \sqcap, \Rightarrow)$ and $(\mathcal{T}, X, \varnothing, \cup, \cap, \Rightarrow)$.

Topological semantics for IPC

Valuation

A valuation o assigns each propositional variable an open set:

$$o : \mathsf{Var} \to \mathcal{T} :: p \mapsto o(p) = [\![p]\!]^o$$

Extend inductively to all formulas, i.e. define $[\![-]\!] : \mathrm{Form}_{\mathsf{IPL}} \to \mathcal{T}$ by

$$[\![\top]\!] = X, \quad [\![\bot]\!] = \varnothing, \quad [\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!], \quad [\![\varphi \vee \psi]\!] = [\![\varphi]\!] \cup [\![\psi]\!],$$

$$[\![\varphi \to \psi]\!] = \mathrm{int}\Big((X \setminus [\![\varphi]\!]) \cup [\![\psi]\!]\Big), \qquad [\![\neg\varphi]\!] = \mathrm{int}\big(X \setminus [\![\varphi]\!]\big).$$

For an open set $U \in \mathcal{T}$,

$$U, o \models \varphi \text{ iff } U \subseteq [\![\varphi]\!].$$

A formula φ is *valid in* $(X, \mathcal{T})$ iff $[\![\varphi]\!] = X$ for every valuation.

Sheaf semantics on spaces = topological semantics

Using the Heyting isomorphism $\mathrm{Sub}_{\mathcal{E}}(1) \cong \mathcal{T}$, every sheaf valuation v in $\mathrm{Sub}_{\mathcal{E}}(1)$ naturally corresponds to a topological valuation o in $\mathcal{T}$.

Then for every $\varphi \in$ IPL and every open stage $U \in \mathcal{T}$,

$$U, o \models_{tp} \varphi \iff U \subseteq [\![\varphi]\!]^o \iff \mathcal{E}, U, v \models_{\mathrm{Sh}} \varphi.$$

For the inductive step for $\vee$, we make use of the local feature of covers. For any $A, B \in \mathcal{T}$, $U \subseteq A \cup B$ iff:

there is an open cover $(U_i)_{i \in I}$ of U s.t. $\forall i \in I$: $U_i \subseteq A$ or $U_i \subseteq B$

which matches the sheaf semantic clause for $\vee$.

In particular, taking $U = X$, for each valuation o (equivalently the induced v), we have $[\![\varphi]\!]^o = X \iff [\![\varphi]\!]^v(X) = \{*\}$; hence $(X, \mathcal{T}) \models \varphi$ iff $\mathrm{Sh}(\mathcal{T}) \models \varphi$.

Sheaf semantics on spaces = topological semantics

Theorem (Tarski)

$\textbf{IPC} = \mathsf{Log}\big(\{\mathrm{Sh}(X, \mathcal{T}) \mid (X, \mathcal{T}) \textit{ is a topological space } \}\big)$

Kripke–Joyal semantics

In any (elementary) topos $\mathcal{E}$, the truth values $\mathrm{Sub}_{\mathcal{E}}(1)$ form a Heyting algebra, so the internal propositional logic of $\mathcal{E}$ is intuitionistic.

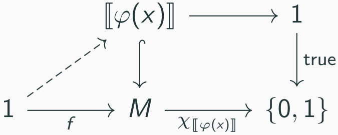
- classical only when $\mathrm{Sub}_{\mathcal{E}}(1)$ is Boolean($\neg\neg = \mathrm{id}$), e.g. when the Grothendieck topology is the dense/double-negation topology, defined by $\mathrm{Cov}(C) = \{R \in \Omega(C) : (\forall f : C' \to C)\,(\exists g : C'' \to C')\,(fg \in R)\}$

Kripke-Joyal semantics

More generally, every elementary topos $\mathcal{E}$ carries an internal first-order intuitionistic logic with logical connectives/quantifiers interpreted by the corresponding categorical constructions (pullbacks, images, exponentials, adjoints). Kripke–Joyal semantics is the standard "external" forcing presentation that lets us read off this internal logic stagewise.

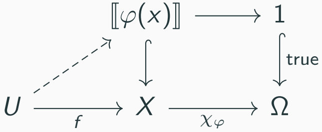
Set structure:

- Domain: a set M
- Relation symbol: subset
- Assignment: elements $\vec{x} \mapsto \vec{a}$
- Formula: "derived" subset $[\![\varphi(x)]\!] \subseteq M$



Topos structure:

- Domain: an object X
- Relation symbol: subobject
- Assignment: generalized elements
- Formula φ: "derived" subobjects



Questions?

M. Artin, A. Grothendieck, and J.-L. Verdier.
Théorie des Topos et Cohomologie Étale des Schémas (SGA 4), volume 269 of *Lecture Notes in Mathematics*.
Springer-Verlag, 1972.

O. Caramello.
Theories, Sites, Toposes.
Oxford University Press, London, England, 2017.

D. Clausen and L. Hesselholt.
Scheme theory.
Lecture Notes, 2022.

J. Lurie.
Categorical logic.
Lecture Notes, 2018.