# TOPOLOGY IN AND VIA LOGIC 2026
# TUTORIAL 2

## Compactness

**Definition 1.** *Let $X$ be a set, and $S \subseteq \mathcal{P}(X)$ closed under intersection, union, complement, and contains $\emptyset$; we say that $S$ is a Boolean algebra. We say that $F \subseteq S$ is an $S$-filter if:*

*(1)* $X \in F$;

*(2) If* $U \in F$ *and* $U \subseteq V$ *where* $V \in S$, *then* $V \in F$;

*(3) If* $U, V \in F$ *then* $U \cap V \in F$.

*Furthermore, we call $F$ a prime $S$-filter if for each $U \in S$, either $U \in F$ or $X - U \in F$.*

*Note:* Below you can use the fact that the Prime Filter Theorem, which we saw in class, holds for any Boolean algebra in the above condition.

**Exercise 1.** *Let $X$ be a topological space. Observe that*

$$\mathsf{Clop}(X) = \{U \subseteq X : U \text{ is clopen}\}$$

*is closed under intersection, union, complement, and contains $\varnothing$. Thus, we can consider the collection of $\mathsf{Clop}(X)$-prime filters, denoted by $X^* = Spec(\mathsf{Clop}(X))$. We give this space a topology by specifying the following basis (you may assume without proof that this, indeed, is a basis for a topology on $X^*$):*

$$\{\varphi(U) : U \in \mathsf{Clop}(X)\} \text{ where } \varphi(U) = \{F \in X^* : U \in F\}.$$

*(1) Show that $X^*$ is always a compact Hausdorff space. Hint: For compactness, given $X^* = \bigcup_{i \in I} \varphi(U_i)$, it might be helpful to consider*

$$\{U \in \mathsf{Clop}(X) \mid U \supseteq U_{i_0}^c \cap \cdots \cap U_{i_n}^c \text{ for some } \{i_0, \ldots, i_n\} \subseteq I\}.$$

*(2) Show that the map $i : X \to X^*$ given by*

$$i(x) \coloneqq \{U \in \mathsf{Clop}(X) : x \in U\}$$

*is well-defined.*

**Definition 2.** *Let $X$ be a normal topological space. We say that $X$ is strongly zero-dimensional if whenever $A, B$ are disjoint closed sets, then there is some clopen set $U$ such that $A \subseteq U$ and $B \subseteq X - U$.*

**Exercise 2.** *Let $X$ be a topological space and $X^*$ be defined as in Exercise 1.*

(3) *Assume that $X$ is a strongly zero-dimensional space, and suppose that $Z$ is some compact Hausdorff space, such that $f : X \to Z$ is a continuous function. Show that for $F \in X^*$ the map*

$$\tilde{f}(F) \coloneqq x_F, \text{ where } x_F \in \bigcap \{\overline{f[U]} : U \in F\}.$$

*is a well-defined continuous map from $X$ to $Z$, and has the property that $\tilde{f} \circ i = f$. (Hint: you can use the following fact without proof: for all distinct $u, v \in Z$, there exists open sets $U \in N(u)$ and $V \in N(v)$ such that $cl(U) \cap cl(V) = \varnothing$.)*

(4) *Conclude that for strongly zero-dimensional spaces we have that $X^* \cong \beta(X)$.*

## Topological Semantics of Modal Logic

McKinsey and Tarski [1] proposed interpretations of the unary modal operator $\Diamond$ as the closure operator $cl$ and the derived set operator $\mathsf{d}$ of a topological space. These interpretations give the topological C-semantics and $\mathsf{d}$-semantics for modal logic.

In this section, we take a closer look at the topological C-semantics of modal logic.

**Definition 3.** *A topological model is a triple $\mathfrak{M} = (X, \tau, \nu)$ where $(X, \tau)$ is a topological space and $\nu : Var \to \mathcal{P}(X)$ a function called a valuation for $X$.*

*A valuation $\nu$ is extended to the set $Fm$ of all modal formulas by the following rules:*

$$\nu(\bot) = \varnothing,\ \nu(\varphi \to \psi) = (X \setminus \nu(\varphi)) \cup \nu(\psi) \text{ and } \nu(\Diamond\varphi) = cl(\nu(\varphi)).$$

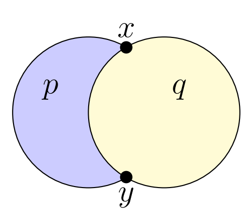
*A formula $\varphi$ is true at $x$ in $\mathfrak{M}$, notation $\mathfrak{M}, x \models \varphi$, if $x \in \nu(\varphi)$. Note that by definition of the operator $cl : \mathcal{P}(X) \to \mathcal{P}(X)$, the following statements hold:*

(a) *$\mathfrak{M}, x \models \Box\varphi$ if and only if there is $U \in N(x)$ such that $\mathfrak{M}, y \models \varphi$ for all $y \in U$;*

(b) *$\mathfrak{M}, x \models \Diamond\varphi$ if and only if for all $U \in N(x)$, there is $y \in U$ such that $\mathfrak{M}, y \models \varphi$.*

**Exercise 3.** *Show that (a) and (b) in Definition 3 hold.*

**Exercise 4.** *Consider the following topological model, where $p$ is true in the blue area and its border line, and $q$ is true in the yellow area and its border line.*




(1) *Draw the region defined by the modal formula $\Diamond p \vee \Box q$.*

(2) *Find modal formulas that define the set $\{x, y\}$.*

Recall that modal logic $\mathsf{S4}$ is defined to be $\mathsf{K} \oplus \{\Diamond\Diamond p \to \Diamond p, p \to \Diamond p\}$.

**Exercise 5.** *Prove that every theorem of* $\mathsf{S4}$ *is valid. That is, given any* $\varphi \in \mathsf{S4}$, $\mathfrak{M}, x \models \varphi$ *for all topological model* $\mathfrak{M} = (X, \tau, \nu)$ *and* $x \in X$.

In fact, the converse of Exercise 5 also holds. Thus we have

**Theorem 4.** $\mathsf{S4}$ *is the modal logic of all topological spaces.*

## More on topological semantics: d-semantics

There are more than one semantics of modal logic based on topological spaces. The one we have seen in class is also called C-semantics. We now introduce the d-semantics as follows:

**Definition 5.** *Let* $\mathcal{X} = (X, \tau)$ *be a topological space and* $x \in X$. *A subset* $Y \subseteq X$ *is an* open neighborhood *of* $x$ *if* $x \in Y \in \tau$. *Let* $N(x)$ *be the set of all open neighborhoods of* $x$. *For every subset* $A \subseteq X$, *let* $\mathsf{d}(A)$ *be the* derived set *of* $A$, *i.e.,*

$$\mathsf{d}(A) = \{x \in X : \forall U \in N(x)(U \cap (A \setminus \{x\}) \neq \varnothing)\}.$$

*A* topological model *is a triple* $\mathcal{M} = (X, \tau, \nu)$ *where* $\mathcal{X} = (X, \tau)$ *is a topological space and* $\nu : \mathsf{Prop} \to \mathcal{P}(X)$ *is a function which is called a* valuation *in* $\mathcal{X}$. *A valuation* $\nu$ *is extended to all modal formulas* $\mathcal{L}$ *as follows:*

$$\nu(\neg\varphi) = X \setminus \nu(\varphi),\ \nu(\varphi \vee \psi) = \nu(\varphi) \cup \nu(\psi) \text{ and } \nu(\Diamond\varphi) = \mathsf{d}(\nu(\varphi)).$$

*For each formula* $\varphi$, $\varphi$ *is d-*true *at* $w$ *in* $\mathcal{M}$ *(notation:* $\mathcal{M}, w \models_{\mathsf{d}} \varphi$*) if* $w \in \nu(\varphi)$. *We say that* $\varphi$ *is d-valid if* $\mathcal{M}, w \models_{\mathsf{d}} \varphi$ *for all topological model* $\mathcal{M}$ *and point* $w$ *in* $\mathcal{M}$.

**Exercise 6.** *Try to understand the d-semantics given above and show:*

*(1)* $\Diamond\Diamond p \to \Diamond p$ *is not* $\mathsf{d}$*-valid.*

*(2)* $\Diamond\Diamond p \to \Diamond p \vee p$ *is* $\mathsf{d}$*-valid.*

## References

[1] McKinsey J. C. C., Tarski A., The algebra of topology, Annals of Mathematics **45**, 141–191 (1944)