TOPOLOGY IN AND VIA LOGIC 2026
TUTORIAL 1

Basic Set Theory

Exercise 1. *The following results are used often in topology: Let X, Y be sets, $f : X \to Y$ a function, $S \subseteq X$, $\{S_i : i \in I\} \subseteq \mathcal{P}(X)$, $T \subseteq Y$ and $\{T_j : j \in J\} \subseteq \mathcal{P}(Y)$. Then*

(1) $f[\bigcup_{i\in I} S_i] = \bigcup_{i\in I} f[S_i]$.

(2) $f[\bigcap_{i\in I} S_i] \subseteq \bigcap_{i\in I} f[S_i]$.

(3) $f^{-1}[\bigcup_{j\in J} T_j] = \bigcup_{j\in J} f^{-1}[T_j]$.

(4) $f^{-1}[\bigcap_{j\in J} T_j] = \bigcap_{j\in J} f^{-1}[T_j]$.

(5) $f[S] \cap T = f[S \cap f^{-1}[T]]$.

Furthermore, if $f[\bigcap_{i\in I} S_i] = \bigcap_{i\in I} f[S_i]$ if f is injective. Prove them.

Basic Topology

Exercise 2. *Recall that the* Euclidean topology τ_{Euc} *on* $\mathbb{R}$ *is defined as follows:*

$$\text{for all } U \subseteq \mathbb{R},\ U \in \tau_{Euc} \text{ if and only if } \forall z \in U \exists x, y \in U (z \in (x, y) \subseteq U).$$

Verify that $(\mathbb{R}, \tau_{Euc})$ is a topological space.

Exercise 3. *Recall that the* Cantor set *is defined to be the set 2^ω of all binary sequences of length ω. Let $2^{<\omega}$ denote the set of all finite binary sequences. For all $s \in 2^{<\omega}$ and $t \in 2^\omega \cup 2^{<\omega}$, we write $s \triangleleft t$ if $t{\restriction}\mathsf{dom}(s) = s$. Intuitively, $s \triangleleft t$ means that s is an initial subsequence of t. For each $s \in 2^{<\omega}$, we define the set $C(s)$ by*

$$C(s) = \{t \in 2^\omega : s \triangleleft t\}.$$

Let $B = \{C(s) : s \in 2^{<\omega}\}$. Verify that there is a unique topology τ_{Can} on the Cantor set for which B is a basis.

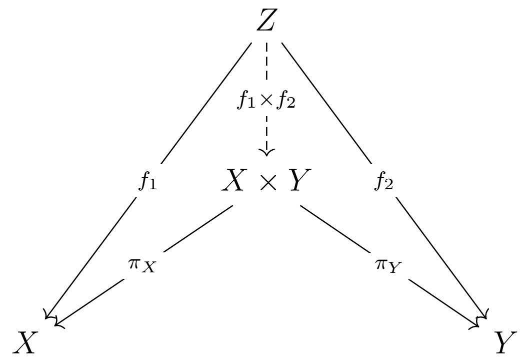
The topological space $(2^\omega, \tau_{Can})$ is called the Cantor space.

Exercise 4. *Let X, Y be topological spaces.*

(1) *Show that the product topology is the coarsest topology on the set $X \times Y$ such that the projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are continuous.*

(2) *Show that for any other topological space Z, if there exist continuous functions $f_1 : Z \to X$ and $f_2 : Z \to Y$, then there exists a unique continuous function $f_1 \times f_2$:*

$Z \to X \times Y$ making the following diagram commute



(3) *Show that this defines the product topology up to homeomorphism: whenever a topological space A together with two continuous functions $\pi_{A,X} : A \to X$ and $\pi_{A,Y} : A \to Y$ satisfy the condition in (2), then there exists a homeomorphism between A and $X \times Y$.* Hint: Given topological spaces X, Y, a continuous map $f : X \to Y$ is a homeomorphism if and only if there is a continuous map $g : Y \to X$ such that $fg = id_Y$ and $gf = id_X$.

Closure, Interior and Neighbourhoods

Definition 1. *Let (X, τ) be a topological space. We say that a set $U \in \mathcal{P}(X)$ is closed if its complement is open, i.e., if $(X \setminus U) \in \tau$.*

Exercise 5. *Let (X, τ) be a topological space and $S \subseteq X$. Show that the following hold:*

(1) *There exists an open set $int(S)$ such that (i) $int(S) \subseteq S$; and (ii) for all open set U, $U \subseteq S$ implies $U \subseteq int(S)$.*

(2) *There exists an closed set $cl(S)$ such that (i) $S \subseteq cl(S)$; and (ii) for all closed set U, $S \subseteq U$ implies $cl(S) \subseteq U$.*

Definition 2. *The sets $int(S)$ and $cl(S)$ in Exercise 5 are called the* interior *and the* closure *of S, respectively. Moreover, we see that a set S is closed if $S = cl(S)$, and open if $S = int(S)$. The operators*

$$int : \mathcal{P}(X) \to \mathcal{P}(X), S \mapsto int(S)$$

and

$$cl : \mathcal{P}(X) \to \mathcal{P}(X), S \mapsto cl(S)$$

are called the topological interior *and* topological closure*, respectively.*

Exercise 6. *Let (X, τ) be a topological space and $A, B \subseteq X$. Prove the following statements:*

(1) *$A \subseteq cl(A)$ and $int(A) \subseteq A$.*

(2) *$cl(cl(A)) = cl(A)$ and $int(int(A)) = int(A)$.*

(3) *$cl(A) = X \setminus (int(X \setminus A))$ and $int(A) = X \setminus (cl(X \setminus A))$.*

(4) *$cl(A) \cup cl(B) = cl(A \cup B)$ and $int(A) \cap int(B) = int(A \cap B)$.*

(5) *If $A \subseteq B$, then $cl(A) \subseteq cl(B)$ and $int(A) \subseteq int(B)$.*

Is $cl(A) \cap cl(B) = cl(A \cap B)$ or $int(A) \cup int(B) = int(A \cup B)$ true in general? Prove your answer.

Definition 3. *Given a topological space (X, τ) and a point $x \in X$, we say that $V \in \mathcal{P}(X)$ is a* neighbourhood of x *if there is an open set U such that $x \in U \subseteq V$.*

Moreover, observe that if a neighbourhood V of a point x is open, the definition simplifies: V is an open neighbourhood of a point x if and only if $x \in V$ and V is open.[1]

Let $N(x)$ denote the set of all open neighbourhoods of x, i.e., $N(x) = \{U \in \tau : x \in U\}$.

Exercise 7. *Suppose (X, τ) is a topological space and $S \subseteq X$. Then for all $x \in X$, the following are equivalent:*

- *x is in the closure of S, i.e., $x \in cl(S)$.*
- *All open neighbourhoods U of x have non-empty intersection with S, i.e.,*

$$\forall U \in N(x)(U \cap S \neq \varnothing).$$

There is a proof of this proposition in the note, but try to prove it yourself first :)

Continuous maps

Exercise 8. *Complete the proof of Proposition 3.1.2 in the notes. That is, prove that the following are equivalent for a map $f : X \to Y$ between topological spaces:*

(1) f is continuous,

(2) For every closed set U in Y, its preimage $f^{-1}[U]$ is closed in X,

(3) For every $x \in X$, whenever $V \subseteq Y$ is an open neighbourhood of $f(x)$, there is an open neighbourhood $U \subseteq X$ of x such that $f[U] \subseteq V$.

Exercise 9. *Prove that for any real numbers a and b such that $a < b$, the interval (a, b) is homeomorphic to the real line $\mathbb{R}$.* Hint: First try to prove it for $(-1, 1)$.

Remark 4. You now have sufficient topological knowledge to understand the jokes made about topologists: doughnuts and coffee mugs. Topology is the study of spaces up to homeomorphism, which means that spaces that can be obtained by this sort of "stretching" behaviour are homeomorphic. But what out! There can be very wild homeomorphisms between spaces.

[1] In the literature, you will sometimes find that a neighbourhood is already required to be open. We do not adopt that convention, but speak of 'open neighbourhoods' when needed.