

## TOPOLOGY IN AND VIA LOGIC 2026 HOMEWORK ASSIGNMENT 2

- Deadline: January 22 at 23:59.
- All exercises are worth the same points.
- The assignment can be completed in teams of up to two people.
- Good luck!

### SEPARATION

**Exercise 1.** Let  $X$  be a  $T_1$ -space.

- (1) Show that if  $X$  is finite, then the topology on it is discrete.
- (2) Show that for each  $x \in X$ ,  $\{x\}$  is closed.
- (3) Show that for each  $x \in X$ , the filter

$$F(x) := \{S \subseteq X : x \in S\}$$

converges uniquely to  $x$ .

- (4) Show that the property in (3) is an alternative definition for  $T_1$ -spaces.

### COMPACTNESS

**Exercise 2.** Give an example of a compact space that is not Hausdorff. *Hint: Consider the Alexandroff topology on linear orders.*

**Exercise 3.** Let  $X$  be a compact Hausdorff space, and  $A$  a closed subspace. Define the equivalence relation  $x \sim y$  if and only if either  $x = y$  or  $x$  and  $y$  are both in  $A$ . Show that the quotient space  $X/\sim$  is compact Hausdorff.

### CONNECTEDNESS

**Exercise 4.** Let  $X$  and  $Y$  be topological spaces.

- (1) Assume that  $f : X \rightarrow Y$  is a homeomorphism. Show that if  $X$  is connected, then so is  $Y$ .
- (2) Show that if  $X$  and  $Y$  are connected, then so is  $X \times Y$ .