Spatial modal logics: some modern perspectives

Nick Bezhanishvili
Institute for Logic, Language and Computation
University of Amsterdam
https://staff.fnwi.uva.nl/n.bezhanishvili/

Thanks to Tenyo for organizing this project!

27 January 2026

Joint work

This is joint work with a lot of colleagues and former students.

- Sam Adam-Day (Oxford), David Gabelaia (Tbilisi), Vincenzo Marra (Milan)
- David Gabelaia (Tbilisi), Gianluca Grilletti (Munich), Vincenzo Ciancia, Diego Latella, Mieke Masink (CNR Pisa).
- David Gabelaia (Tbilisi), David Fernandez Duque (Barcelona), Laura Bussi and Vincenzo Ciancia (CNR Pisa).

Spatial logic

- In spatial logic a central role is played by the topological semantics of intuitionistic and modal logics.

Spatial logic

- In spatial logic a central role is played by the topological semantics of intuitionistic and modal logics.
- I will discuss a variant of this semantics that connects modal logic with polyhedral geometry.

Spatial logic

- In spatial logic a central role is played by the topological semantics of intuitionistic and modal logics.
- I will discuss a variant of this semantics that connects modal logic with polyhedral geometry.
- We call this new topic polyhedral modal logic.

Spatial logic

- In spatial logic a central role is played by the topological semantics of intuitionistic and modal logics.
- I will discuss a variant of this semantics that connects modal logic with polyhedral geometry.
- We call this new topic polyhedral modal logic.
- I will also review some of the applications of this approach.

Modal logic

- Modal logic is an expressive and yet decidable fragment of first-order logic.

Modal logic

- Modal logic is an expressive and yet decidable fragment of first-order logic.
- This makes it a perfect language to reason about relational structures.

Modal logic

- Modal logic is an expressive and yet decidable fragment of first-order logic.
- This makes it a perfect language to reason about relational structures.
- In the form of modal checking it has been successfully used in specifying and verifying correctness of programs.

Modal logic

- Modal logic is an expressive and yet decidable fragment of first-order logic.
- This makes it a perfect language to reason about relational structures.
- In the form of modal checking it has been successfully used in specifying and verifying correctness of programs.
- We will view modal logic as a bridge between spatial and relational structures.

Modal logic

- Modal logic is an expressive and yet decidable fragment of first-order logic.
- This makes it a perfect language to reason about relational structures.
- In the form of modal checking it has been successfully used in specifying and verifying correctness of programs.
- We will view modal logic as a bridge between spatial and relational structures.
- I will try to illustrate that modal logic also provides a powerful tool for spatial reasoning.

Part 1: Topological semantics of modal logic

Some history

- One of the first semantics of modal logic is topological, introduced some 20 years before Kripke semantics.

Some history

- One of the first semantics of modal logic is topological, introduced some 20 years before Kripke semantics.
- The pioneers of topological semantics were Tarski (1938), Tsao-Chen (1938), McKinsey (1941), and McKinsey and Tarski (1944).

Some history

- One of the first semantics of modal logic is topological, introduced some 20 years before Kripke semantics.
- The pioneers of topological semantics were Tarski (1938), Tsao-Chen (1938), McKinsey (1941), and McKinsey and Tarski (1944).
- They were influenced by the work of Kuratowski (1922) who axiomatized topological spaces by means of closure operators.

Kuratowski's axioms and **S4**

Kuratowski's axioms closely resemble the axioms of Lewis' modal system **S4**:

Kuratowski's axioms and **S4**

Kuratowski's axioms closely resemble the axioms of Lewis' modal system **S4**:

$\mathbf{c}\varnothing = \varnothing$	$\Diamond\bot \leftrightarrow \bot$
$\mathbf{c}(A \cup B) = \mathbf{c}A \cup \mathbf{c}B$	$\Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q$
$A \subseteq \mathbf{c}A$	$p \to \Diamond p$
$\mathbf{cc}A \subseteq \mathbf{c}A$	$\Diamond\Diamond p \to \Diamond p$

Kuratowski's axioms and **S4**

Kuratowski's axioms closely resemble the axioms of Lewis' modal system **S4**:

$\mathbf{c}\varnothing = \varnothing$	$\Diamond\bot \leftrightarrow \bot$
$\mathbf{c}(A \cup B) = \mathbf{c}A \cup \mathbf{c}B$	$\Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q$
$A \subseteq \mathbf{c}A$	$p \to \Diamond p$
$\mathbf{cc}A \subseteq \mathbf{c}A$	$\Diamond\Diamond p \to \Diamond p$

$\mathbf{i}X = X$	$\Box\top \leftrightarrow \top$
$\mathbf{i}(A \cap B) = \mathbf{i}A \cap \mathbf{i}B$	$\Box(p \wedge q) \leftrightarrow \Box p \wedge \Box q$
$\mathbf{i}A \subseteq A$	$\Box p \to p$
$\mathbf{i}A \subseteq \mathbf{ii}A$	$\Box p \to \Box\Box p$

S4 as the logic of topological spaces

- For a formula φ let $[\![\varphi]\!] = \{x \in X \mid x \models \varphi\}$.

$\mathbf{S4}$ as the logic of topological spaces

- For a formula φ let $[\![\varphi]\!] = \{x \in X \mid x \models \varphi\}$.
- Then $x \models \Box\varphi$ iff $x \in \mathbf{i}[\![\varphi]\!]$ and $x \models \Diamond\varphi$ iff $x \in \mathbf{c}[\![\varphi]\!]$.

S4 as the logic of topological spaces

- For a formula φ let $[\![\varphi]\!] = \{x \in X \mid x \models \varphi\}$.
- Then $x \models \Box\varphi$ iff $x \in \mathbf{i}[\![\varphi]\!]$ and $x \models \Diamond\varphi$ iff $x \in \mathbf{c}[\![\varphi]\!]$.
- Thus, $\Box$ is interpreted as interior and $\Diamond$ as closure.

S4 as the logic of topological spaces

- For a formula φ let $[\![\varphi]\!] = \{x \in X \mid x \models \varphi\}$.
- Then $x \models \Box\varphi$ iff $x \in \mathbf{i}[\![\varphi]\!]$ and $x \models \Diamond\varphi$ iff $x \in \mathbf{c}[\![\varphi]\!]$.
- Thus, $\Box$ is interpreted as interior and $\Diamond$ as closure.
- Consequently, each topological space validates **S4**.

S4 as the logic of topological spaces

- For a formula φ let $[\![\varphi]\!] = \{x \in X \mid x \models \varphi\}$.
- Then $x \models \Box\varphi$ iff $x \in \mathbf{i}[\![\varphi]\!]$ and $x \models \Diamond\varphi$ iff $x \in \mathbf{c}[\![\varphi]\!]$.
- Thus, $\Box$ is interpreted as interior and $\Diamond$ as closure.
- Consequently, each topological space validates **S4**.
- The converse is also true, and hence **S4** is the logic of all topological spaces when $\Box$ is interpreted as interior and $\Diamond$ as closure.

S4 as the logic of topological spaces

- For a formula φ let $[\![\varphi]\!] = \{x \in X \mid x \models \varphi\}$.
- Then $x \models \Box\varphi$ iff $x \in \mathbf{i}[\![\varphi]\!]$ and $x \models \Diamond\varphi$ iff $x \in \mathbf{c}[\![\varphi]\!]$.
- Thus, $\Box$ is interpreted as interior and $\Diamond$ as closure.
- Consequently, each topological space validates **S4**.
- The converse is also true, and hence **S4** is the logic of all topological spaces when $\Box$ is interpreted as interior and $\Diamond$ as closure.
- But much stronger results hold...

McKinsey-Tarski Theorem

A topological space is dense-in-itself if every point is a limit point.

McKinsey-Tarski Theorem

A topological space is dense-in-itself if every point is a limit point.

Theorem

(McKinsey-Tarski, 1944) **S4** is the logic of an arbitrary (nonempty) dense-in-itself metric space.

McKinsey-Tarski Theorem

A topological space is dense-in-itself if every point is a limit point.

Theorem

(McKinsey-Tarski, 1944) **S4** is the logic of an arbitrary (nonempty) dense-in-itself metric space.

Remark

The original McKinsey-Tarski result had an additional assumption that the space is separable.

McKinsey-Tarski Theorem

A topological space is dense-in-itself if every point is a limit point.

Theorem

(McKinsey-Tarski, 1944) **S4** is the logic of an arbitrary (nonempty) dense-in-itself metric space.

Remark

The original McKinsey-Tarski result had an additional assumption that the space is separable. In their 1963 book Rasiowa and Sikorski showed that this additional condition can be dropped.

McKinsey-Tarski Theorem

A topological space is dense-in-itself if every point is a limit point.

Theorem

(McKinsey-Tarski, 1944) **S4** is the logic of an arbitrary (nonempty) dense-in-itself metric space.

Remark

The original McKinsey-Tarski result had an additional assumption that the space is separable. In their 1963 book Rasiowa and Sikorski showed that this additional condition can be dropped. Their proof uses the Axiom of Choice.

How to prove the McKinsey-Tarski Theorem

- It is a well-known fact in modal logic that **S4** has the finite model property, meaning that each non-theorem is refuted on a finite Kripke frame, where the binary relation is reflexive and transitive.

How to prove the McKinsey-Tarski Theorem

- It is a well-known fact in modal logic that **S4** has the finite model property, meaning that each non-theorem is refuted on a finite Kripke frame, where the binary relation is reflexive and transitive. Such frames are called **S4**-frames.

How to prove the McKinsey-Tarski Theorem

- It is a well-known fact in modal logic that **S4** has the finite model property, meaning that each non-theorem is refuted on a finite Kripke frame, where the binary relation is reflexive and transitive. Such frames are called **S4**-frames.
- Since refuting a formula at a point x of an **S4**-frame $\mathfrak{F}$ only requires the points from $R[x]$, we may assume that $\mathfrak{F}$ is rooted, meaning that there is a point, called a root, such that every point is seen from it.

How to prove the McKinsey-Tarski Theorem

- It is a well-known fact in modal logic that **S4** has the finite model property, meaning that each non-theorem is refuted on a finite Kripke frame, where the binary relation is reflexive and transitive. Such frames are called **S4**-frames.
- Since refuting a formula at a point x of an **S4**-frame $\mathfrak{F}$ only requires the points from $R[x]$, we may assume that $\mathfrak{F}$ is rooted, meaning that there is a point, called a root, such that every point is seen from it.
- Given a dense-in-itself metric space X, the key is to transfer each such finite refutation to X. This can be done by defining an onto map $f : X \to \mathfrak{F}$ that behaves like a p-morphism or functional bisimulation.

How to prove the McKinsey-Tarski Theorem

- Such maps are called interior maps in topology, and they satisfy $\mathbf{i}f^{-1}(A) = f^{-1}(\mathbf{i}A)$ or equivalently $\mathbf{c}f^{-1}(A) = f^{-1}(\mathbf{c}A)$.

How to prove the McKinsey-Tarski Theorem

- Such maps are called interior maps in topology, and they satisfy $\mathbf{i}f^{-1}(A) = f^{-1}(\mathbf{i}A)$ or equivalently $\mathbf{c}f^{-1}(A) = f^{-1}(\mathbf{c}A)$.
- Interior maps are exactly the maps that are continuous (the inverse image of an open set is open) and open (the direct image of an open set is open).

How to prove the McKinsey-Tarski Theorem

- Such maps are called interior maps in topology, and they satisfy $\mathbf{i}f^{-1}(A) = f^{-1}(\mathbf{i}A)$ or equivalently $\mathbf{c}f^{-1}(A) = f^{-1}(\mathbf{c}A)$.
- Interior maps are exactly the maps that are continuous (the inverse image of an open set is open) and open (the direct image of an open set is open).

- Constructing such a map from X onto an arbitrary finite rooted **S4**-frame is the main challenge in proving the McKinsey-Tarski theorem.

How to prove the McKinsey-Tarski Theorem

- Such maps are called interior maps in topology, and they satisfy $\mathbf{i}f^{-1}(A) = f^{-1}(\mathbf{i}A)$ or equivalently $\mathbf{c}f^{-1}(A) = f^{-1}(\mathbf{c}A)$.
- Interior maps are exactly the maps that are continuous (the inverse image of an open set is open) and open (the direct image of an open set is open).
- Constructing such a map from X onto an arbitrary finite rooted **S4**-frame is the main challenge in proving the McKinsey-Tarski theorem.
- But as soon as such a map is constructed, the rest of the proof is easy: each non-theorem φ of **S4** is refuted on a finite rooted **S4**-frame $\mathfrak{F}$.

How to prove the McKinsey-Tarski Theorem

- Such maps are called interior maps in topology, and they satisfy $\mathbf{i}f^{-1}(A) = f^{-1}(\mathbf{i}A)$ or equivalently $\mathbf{c}f^{-1}(A) = f^{-1}(\mathbf{c}A)$.
- Interior maps are exactly the maps that are continuous (the inverse image of an open set is open) and open (the direct image of an open set is open).
- Constructing such a map from X onto an arbitrary finite rooted **S4**-frame is the main challenge in proving the McKinsey-Tarski theorem.
- But as soon as such a map is constructed, the rest of the proof is easy: each non-theorem φ of **S4** is refuted on a finite rooted **S4**-frame $\mathfrak{F}$. Utilizing $f : X \to \mathfrak{F}$, we can pull the refutation of φ from $\mathfrak{F}$ to X.

How to prove the McKinsey-Tarski Theorem

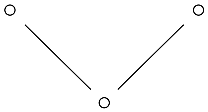
- Such maps are called interior maps in topology, and they satisfy $\mathbf{i}f^{-1}(A) = f^{-1}(\mathbf{i}A)$ or equivalently $\mathbf{c}f^{-1}(A) = f^{-1}(\mathbf{c}A)$.
- Interior maps are exactly the maps that are continuous (the inverse image of an open set is open) and open (the direct image of an open set is open).

- Constructing such a map from X onto an arbitrary finite rooted **S4**-frame is the main challenge in proving the McKinsey-Tarski theorem.

- But as soon as such a map is constructed, the rest of the proof is easy: each non-theorem φ of **S4** is refuted on a finite rooted **S4**-frame $\mathfrak{F}$. Utilizing $f : X \to \mathfrak{F}$, we can pull the refutation of φ from $\mathfrak{F}$ to X. Thus, each non-theorem of **S4** is refuted on X, yielding completeness of **S4** with respect to X.

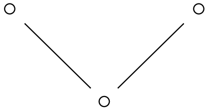
Easy example

Let X be the real line $\mathbb{R}$ and $\mathfrak{F}$ the two-fork



Easy example

Let X be the real line $\mathbb{R}$ and $\mathfrak{F}$ the two-fork



Define $f : \mathbb{R} \to \mathfrak{F}$ by sending 0 to the root, the negatives to one maximal node, and the positives to the other maximal node.

How to handle clusters

Define f from $\mathbb{R}$ onto the two-point cluster

$$\circ \longleftrightarrow \circ$$

by sending the rationals to one node and the irrationals to the other.

How to handle clusters

Define f from $\mathbb{R}$ onto the two-point cluster

$$\circ \longleftrightarrow \circ$$

by sending the rationals to one node and the irrationals to the other.

Intermezzo: Modal dimension

Modal dimension

Definition (G.B., N.B., J. Lucero-Bryan, J. van Mill, 2017)
Modal dimension for topological spaces is defined recursively as follows:

$$\begin{array}{lll}
\mathrm{mdim}(X) = -1 & \text{if} & X = \varnothing, \\
\mathrm{mdim}(X) \leqslant n & \text{if} & \mathrm{mdim}(D) \leqslant n-1 \text{ for } D \text{ nowhere dense in } X, \\
\mathrm{mdim}(X) = n & \text{if} & \mathrm{mdim}(X) \leqslant n \text{ and } \mathrm{mdim}(X) \nleqslant n-1, \\
\mathrm{mdim}(X) = \infty & \text{if} & \mathrm{mdim}(X) \nleqslant n \text{ for any } n = -1, 0, 1, 2, \ldots.
\end{array}$$

Modal dimension

For $n \geqslant 0$, consider the formulas:

$$\begin{aligned} \mathsf{bd}_0 &= \bot, \\ \mathsf{bd}_{n+1} &= \Diamond(\Box p_{n+1} \wedge \neg \mathsf{bd}_n) \to p_{n+1}. \end{aligned}$$

Modal dimension

Let $\mathfrak{F}_n$ be the n-element chain.

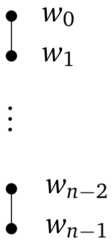


Figure: The n-element chain.

Modal dimension

Theorem (G.B., N.B., J. Lucero-Bryan, J. van Mill, 2017). Let X be a topological space and $n \geqslant 1$. The following are equivalent:

Modal dimension

Theorem (G.B., N.B., J. Lucero-Bryan, J. van Mill, 2017). Let X be a topological space and $n \geqslant 1$. The following are equivalent:

1. $\mathrm{mdim}(X) \leqslant n - 1$.
2. $X \vDash \mathsf{bd}_n$.
3. $\mathfrak{F}_{n+1}$ is not a continuous and open image of X.

Modal dimension

Theorem (G.B., N.B., J. Lucero-Bryan, J. van Mill, 2017). Let X be a topological space and $n \geqslant 1$. The following are equivalent:

1. $\mathrm{mdim}(X) \leqslant n - 1$.
2. $X \vDash \mathsf{bd}_n$.
3. $\mathfrak{F}_{n+1}$ is not a continuous and open image of X.

Fact. $\mathrm{mdim}(\mathbb{Q}) = \mathrm{mdim}(\mathbb{R}) = \mathrm{mdim}(\mathbb{R}^n) = \infty$.

Modal dimension

Theorem (G.B., N.B., J. Lucero-Bryan, J. van Mill, 2017). Let X be a topological space and $n \geqslant 1$. The following are equivalent:

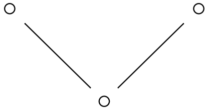
1. $\mathrm{mdim}(X) \leqslant n - 1$.
2. $X \vDash \mathsf{bd}_n$.
3. $\mathfrak{F}_{n+1}$ is not a continuous and open image of X.

Fact. $\mathrm{mdim}(\mathbb{Q}) = \mathrm{mdim}(\mathbb{R}) = \mathrm{mdim}(\mathbb{R}^n) = \infty$.

Theorem (G.B., N.B., J. Lucero-Bryan, J. van Mill, 2021). A modal logic of the diamond frame is complete wrt a normal space iff there is a measurable cardinal.

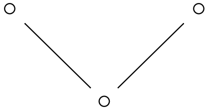
The logic of intervals

If we consider the smaller Boolean algebra generated by the open intervals of $\mathbb{R}$, then we can only pick up the two-fork



The logic of intervals

If we consider the smaller Boolean algebra generated by the open intervals of $\mathbb{R}$, then we can only pick up the two-fork



Theorem (Aiello, van Benthem, G. Bezhanishvili, 2003)

The logic of the two-fork is the logic of the Boolean algebra generated by the open intervals of $\mathbb{R}$.

Euclidean hierarchy

McKinsey and Tarski theorem implies that modal logic of each Euclidean space is **S4**.

Euclidean hierarchy

McKinsey and Tarski theorem implies that modal logic of each Euclidean space is **S4**.

However, we can distinguish the logics of Euclidean spaces of different dimensions by restricting the valuation to special subsets.

Euclidean hierarchy

McKinsey and Tarski theorem implies that modal logic of each Euclidean space is **S4**.

However, we can distinguish the logics of Euclidean spaces of different dimensions by restricting the valuation to special subsets.

Theorem (van Benthem, G. Bezhanishvili, Gehrke, 2003)

More generally, there is a decreasing sequence of logics $\mathbf{L}_n$ $(n \geqslant 1)$ such that each $\mathbf{L}_n$ is the logic of the Boolean algebra generated by the open hypercubes in $\mathbb{R}^n$. Each $\mathbf{L}_n$ is the logic of the n-product of the two-fork.

Euclidean hierarchy

McKinsey and Tarski theorem implies that modal logic of each Euclidean space is **S4**.

However, we can distinguish the logics of Euclidean spaces of different dimensions by restricting the valuation to special subsets.

Theorem (van Benthem, G. Bezhanishvili, Gehrke, 2003)

More generally, there is a decreasing sequence of logics $\mathbf{L}_n$ $(n \geqslant 1)$ such that each $\mathbf{L}_n$ is the logic of the Boolean algebra generated by the open hypercubes in $\mathbb{R}^n$. Each $\mathbf{L}_n$ is the logic of the n-product of the two-fork.

This is the beginning of our story...

Euclidean hierarchy

McKinsey and Tarski theorem implies that modal logic of each Euclidean space is **S4**.

However, we can distinguish the logics of Euclidean spaces of different dimensions by restricting the valuation to special subsets.

Theorem (van Benthem, G. Bezhanishvili, Gehrke, 2003)

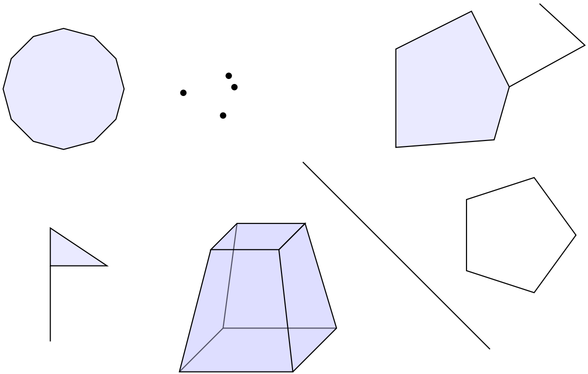
More generally, there is a decreasing sequence of logics $\mathbf{L}_n$ $(n \geqslant 1)$ such that each $\mathbf{L}_n$ is the logic of the Boolean algebra generated by the open hypercubes in $\mathbb{R}^n$. Each $\mathbf{L}_n$ is the logic of the n-product of the two-fork.

This is the beginning of our story...

This is joint work with Sam Adam-Day (Oxford), David Gabelaia (Tbilisi) and Vincenzo Marra (Milan).

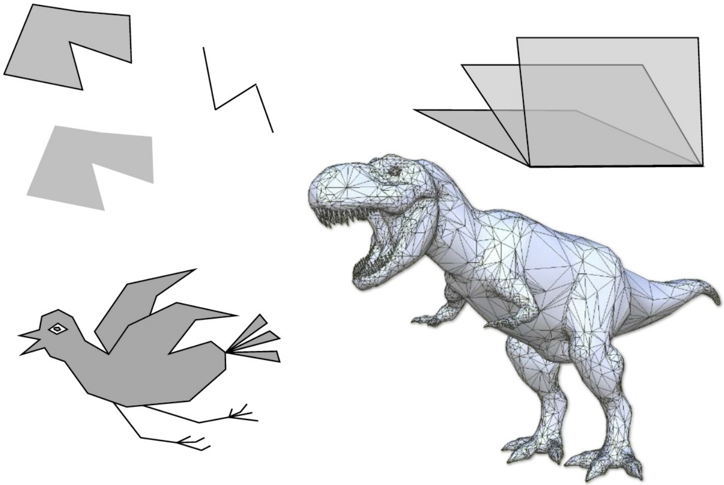
Part 2: Polyhedral semantics of modal logic

Polyhedra



- Polyhedra can be of any dimension, and need not be convex nor connected.
- Formally: Boolean combination of convex hulls of finite sets.
- Alternatively they are solution sets of linear inequalities.

Polyhedra



The Boolean algebra Sub(P)

Theorem

The set of subpolyhedra Sub(P) of a polyhedron P forms a Boolean algebra closed under interior and closure.

The Boolean algebra Sub(P)

Theorem

The set of subpolyhedra Sub(P) of a polyhedron P forms a Boolean algebra closed under interior and closure.

The Boolean algebra Sub(P)

Theorem

The set of subpolyhedra Sub(P) of a polyhedron P forms a Boolean algebra closed under interior and closure.

So we arrive at a polyhedral semantics for modal and intuitionistic logics.

Polyhedral semantcis

Let P be a polyhedron.

Polyhedral semantcis

Let P be a polyhedron.

A valuation is a map $V : \mathsf{Prop} \to \mathrm{Sub}(P)$.

Polyhedral semantcis

Let P be a polyhedron.

A valuation is a map $V : \mathsf{Prop} \to \mathsf{Sub}(P)$.

This valuation is extended to all modal formulas in a standard way:

$$V(\Box\varphi) = \mathbf{i}(V(\varphi)), \;\; V(\Diamond\varphi) = \mathbf{c}(V(\varphi)).$$

Polyhedral semantcis

Let P be a polyhedron.

A valuation is a map $V : \mathsf{Prop} \to \mathrm{Sub}(P)$.

This valuation is extended to all modal formulas in a standard way:

$$V(\Box\varphi) = \mathbf{i}(V(\varphi)), \quad V(\Diamond\varphi) = \mathbf{c}(V(\varphi)).$$

Then P validates φ (written: $P \models \varphi$) if $V(\varphi) = P$ under any valuation V.

Polyhedral semantcis

Let P be a polyhedron.

A valuation is a map $V : \mathsf{Prop} \to \mathrm{Sub}(P)$.

This valuation is extended to all modal formulas in a standard way:

$$V(\Box\varphi) = \mathbf{i}(V(\varphi)), \quad V(\Diamond\varphi) = \mathbf{c}(V(\varphi)).$$

Then P validates φ (written: $P \models \varphi$) if $V(\varphi) = P$ under any valuation V.

In other words, $P \models \varphi$ if φ is valid in the algebra $\mathrm{Sub}(P)$.

Polyhedral semantcis

Let P be a polyhedron.

A valuation is a map $V : \mathsf{Prop} \to \mathrm{Sub}(P)$.

This valuation is extended to all modal formulas in a standard way:

$$V(\Box\varphi) = \mathbf{i}(V(\varphi)), \;\; V(\Diamond\varphi) = \mathbf{c}(V(\varphi)).$$

Then P validates φ (written: $P \models \varphi$) if $V(\varphi) = P$ under any valuation V.

In other words, $P \models \varphi$ if φ is valid in the algebra $\mathrm{Sub}(P)$.

Our aim is to investigate this semantics.

Polyhedral Completeness: Two Approaches

Definition

A logic is polyhedrally complete (poly-complete) if it is the logic of some class of polyhedra.

Polyhedral Completeness: Two Approaches

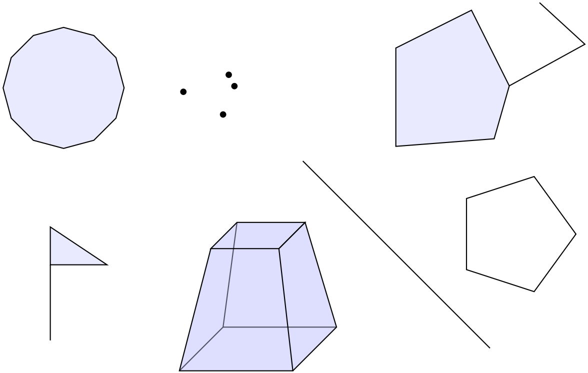
Definition

A logic is polyhedrally complete (poly-complete) if it is the logic of some class of polyhedra.

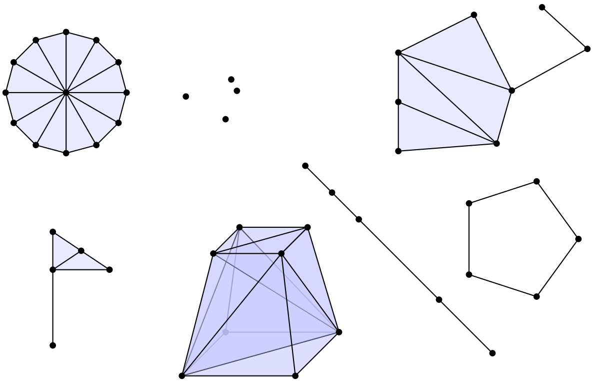
We investigate the phenomenon of poly-completeness from two directions.

1. Which logics are poly-complete?
2. Given a class of polyhedra, what is its logic?
3. Path toward applications.

Polyhedra



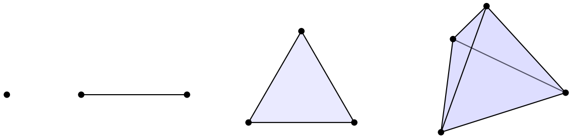
Triangulations I



Intuition: triangulations break polyhedra up into simple shapes.

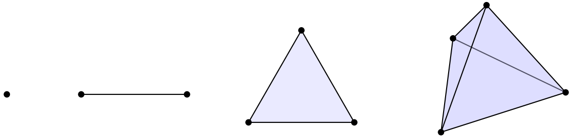
Triangulations II

- Simplices are the most basic polyhedra of each dimension.
- Points, line segments, triangles, tetrahedra, pentachora, etc.

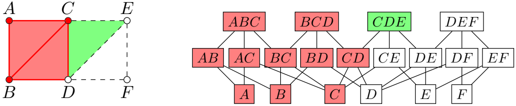


Triangulations II

- Simplices are the most basic polyhedra of each dimension.
- Points, line segments, triangles, tetrahedra, pentachora, etc.



- A triangulation is a splitting up of a polyhedron into finitely many simplices.
- Represented as a poset $(\Sigma, \preccurlyeq)$ of simplices, where $\sigma \preccurlyeq \tau$ means that σ is a face of τ.
- Its underlying polyhedron is $|\Sigma| := \bigcup \Sigma$.
- Every polyhedron admits a triangulation.



(a) Polyhedral model $\mathcal{X}$

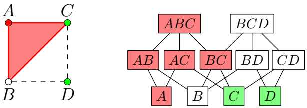
(b) Kripke model $\mathcal{M}(\mathcal{X})$

Figure 7: The polyhedral model $\mathcal{X}$ of Figure 17 (7a) and its corresponding Kripke model $\mathcal{M}(\mathcal{X})$ (7b). We indicate a cell by the set of the vertices of the corresponding simplex. The accessibility relation $\widetilde{\preceq}$ is represented via its Hasse diagram (reflexive and transitive edges are omitted). The atomic propositions g and r are indicated in green and red respectively.

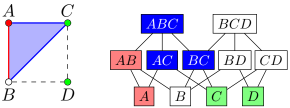
- $\widetilde{\mathrm{K}}$ is the simplicial partition of $|\mathrm{K}|$ generated by K, as Defined in Lemma 2.4,
- $\widetilde{\preceq} \subseteq \widetilde{\mathrm{K}} \times \widetilde{\mathrm{K}}$ with $\widetilde{\sigma}_1 \,\widetilde{\preceq}\, \widetilde{\sigma}_2$ iff $\sigma_1 \preceq \sigma_2$, and
- $\widetilde{\sigma} \in \widetilde{V}(p)$ iff $\widetilde{\sigma} \subseteq V(p)$

where $\preceq$ is the face relation of the simplicial complex K.

Notice that, since $\preceq$ is reflexive, anti-symmetric and transitive, then so is $\widetilde{\preceq}$. An example of a 2D polyhedral model together with its corresponding Kripke model $\mathcal{M}(\mathcal{X})$ is depicted in

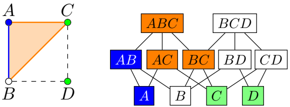


(a) A polyhedral model $\mathcal{X}$ and its corresponding Kripke model $\mathcal{M}(\mathcal{X})$.



$\texttt{frontier} \quad = \{AC, BC, ABC\}$

(b) Initialization of **frontier** (lines 4-5).



$\texttt{frontier} \quad = \{A, AB\}$

$\texttt{flooded} \quad = \{AC, BC, ABC\}$

Triangulation posets

Triangulation posets

Theorem

The logic of a polyhedron is the logic of its triangulations.

Triangulation posets

Theorem

The logic of a polyhedron is the logic of its triangulations.

Proof sketch: If $P \not\models \varphi$, then there is a finite triangulation Σ of P that refutes φ.

Triangulation posets

Theorem

The logic of a polyhedron is the logic of its triangulations.

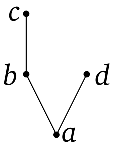
Proof sketch: If $P \not\models \varphi$, then there is a finite triangulation Σ of P that refutes φ.

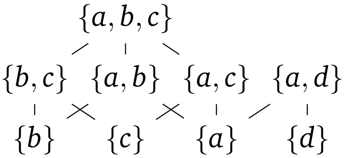
The poset of triangulations contains all the logical information of P.

The Nerve

Definition (Alexandroff's nerve)

The nerve, $\mathcal{N}(F)$, of a finite poset F is the set of all non-empty chains in F, ordered by inclusion.

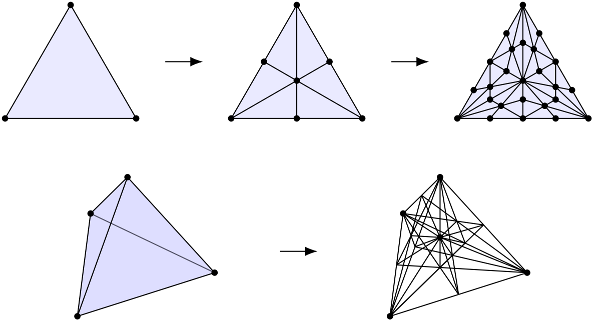




There is always a p-morphism $\mathcal{N}(F) \to F$.

Barycentric Subdivision

Given a triangulation Σ, construct its barycentric subdivision Σ' by putting a new point in the middle of each simplex, and forming a new triangulation around it.



$$\Sigma' \cong \mathcal{N}(\Sigma) \text{ as posets.}$$

Barycentric Subdivision and the Nerve Criterion

Theorem (Nerve Criterion) A logic $\mathcal{L}$ is poly-complete if and only if it is the logic of a class $\mathbf{C}$ of finite frames closed under $\mathcal{N}$.

Barycentric Subdivision and the Nerve Criterion

Theorem (Nerve Criterion) A logic $\mathcal{L}$ is poly-complete if and only if it is the logic of a class **C** of finite frames closed under $\mathcal{N}$.

- This is about barycentric subdivision.
- Let $\Sigma^{(n)}$ be the nth iterated barycentric subdivision of Σ.
- Intuition: $(\Sigma^{(n)})_{n\in\mathbb{N}}$ captures everything (logical) about $P = |\Sigma|$.

Nerves

Theorem (Goes back to Alexandroff). For each finite frame F there is a polyhedron P and a triangulation of P such that the face poset Σ of P is $\mathcal{N}(F)$.

Consequences

Theorem

- The logics S4.Grz and BD_n are poly-complete for every $n \in \mathbb{N}$.
- The logics S4.Grz.2, S4.Grz.3, S4.Grz.3_n, BW_n, BTW_n and BC_n are poly-incomplete.
- Moreover, there are continuum-many logics which are poly-incomplete and have the FMP (stable modal logics).

Consequences

Theorem

- The logics S4.Grz and BD_n are poly-complete for every $n \in \mathbb{N}$.
- The logics S4.Grz.2, S4.Grz.3, $S4.Grz.3_n$, BW_n, BTW_n and BC_n are poly-incomplete.
- Moreover, there are continuum-many logics which are poly-incomplete and have the FMP (stable modal logics).

The key idea: (1) use the Nerve Criterion and note that S4.Grz is the logic of all finite posets and the nerve construction does not increase the height of a poset.

Consequences

Theorem

- The logics S4.Grz and BD_n are poly-complete for every $n \in \mathbb{N}$.
- The logics S4.Grz.2, S4.Grz.3, $S4.Grz.3_n$, BW_n, BTW_n and BC_n are poly-incomplete.
- Moreover, there are continuum-many logics which are poly-incomplete and have the FMP (stable modal logics).

The key idea: (1) use the Nerve Criterion and note that S4.Grz is the logic of all finite posets and the nerve construction does not increase the height of a poset.

(2), (3) Note that repeatedly applying $\mathcal{N}$ produces wider and wider frames.

Consequences

Theorem

- The logics S4.Grz and BD_n are poly-complete for every $n \in \mathbb{N}$.
- The logics S4.Grz.2, S4.Grz.3, $S4.Grz.3_n$, BW_n, BTW_n and BC_n are poly-incomplete.
- Moreover, there are continuum-many logics which are poly-incomplete and have the FMP (stable modal logics).

The key idea: (1) use the Nerve Criterion and note that S4.Grz is the logic of all finite posets and the nerve construction does not increase the height of a poset.

(2), (3) Note that repeatedly applying $\mathcal{N}$ produces wider and wider frames. Are there other poly-complete logics?

Polyhedral Completeness in Intermediate and Modal Logics

MSc Thesis *(Afstudeerscriptie)*

written by

Sam Adam-Day
(born 15th September, 1993 in Bath, United Kingdom)

under the supervision of **Dr. Nick Bezhanishvili**, and submitted to the Board of Examiners in partial fulfillment of the requirements for the degree of

MSc in Logic

at the *Universiteit van Amsterdam*.

Date of the public defense:	**Members of the Thesis Committee:**
2nd July, 2019	Dr. Alexandru Baltag
	Prof. Johan van Benthem
	Dr. Nick Bezhanishvili (supervisor)
	Prof. Yde Venema (chair)

Polyhedral semantics of modal logic

MSc Thesis *(Afstudeerscriptie)*

written by

P. Maurice Dekker
(born 25 September 1998 in Zaandijk, Netherlands)

under the supervision of **Nick Bezhanishvili** and **David Gabelaia**, and submitted to the Examinations Board in partial fulfillment of the requirements for the degree of

MSc in Logic

at the *Universiteit van Amsterdam*.

Date of the public defense:	**Members of the Thesis Committee:**
30 June 2023	Lev Beklemishev
	Benno van den Berg (chair)
	Nick Bezhanishvili (supervisor)
	David Gabelaia (supervisor)
	Aybüke Özgün

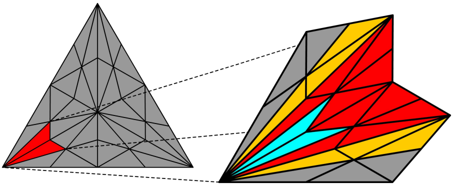


Figure 7.1: example of Lemma 7.8 when $n = 2$ (left) and $n = 3$ (right)

7.2 Efficiently bounded triangulations in $\mathbb{R}^3$

In this section we prove *EffBound*$_\varnothing(\mathfrak{p})$ for a particular set $\mathfrak{p} \subseteq \mathbf{plhdr}_2$ (Theorem 7.12). To prove such a result, we need to be able to build Kripke models whose underlying frames are iterated barycentric subdivisions of triangulations of polyhedra in $\mathfrak{p}$. Since twodimensional complexes are for an important part built from triangles, it may not be surprising that we have some technical lemmas about the behaviour of iterated barycentric subdivisions in relation to triangles.

The first lemma describes how to subdivide a triangle into areas that some chosen vertex of the triangle is the only place where more than two areas meet. Logically, this is interesting for the local structure at the chosen vertex, without the local structure becoming too complicated elsewhere.

Lemma **7.8**. Let τ be a triangle, $\mathbf{x} \in \mathrm{vtc}(\tau)$, $n \geq 1$ and $1 \leq m \leq 2^{n-1}$. Then there exists a partition $\mathscr{T} = \{T_0, \ldots, T_{m-1}\}$ of the set of triangles in $\mathrm{fac}(\tau)^{+n}$ such that:

- $\#\mathscr{T} = m$;
- for each $T \in \mathscr{T}$ there exists $\tau \in T$ with $\mathbf{x} \in \tau$;
- whenever a triangle in T_i and a triangle in T_j intersect (other than at $\mathbf{x}$), we have $j \in \{i - 1, i, i+1\}$;
- all triangles in $\mathrm{fac}(\tau)^{+n}$ that intersect $\partial\tau \setminus \{\mathbf{x}\}$ are in T_{m-1}.

We omit a proof, since everything happens within the triangle τ and is therefore easy to visualize. Some examples are depicted in Figure 7.1. The next lemma describes how to "separate" two onedimensional polyhedra that lie within some twodimensional polyhedron. It does so by subdividing the twodimensional polyhedron into a list of areas such that only the first area touches

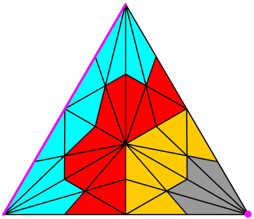


Figure 7.2: example of Lemma 7.9 when $n = 2$ with Λ_0 and Λ_1 in pink

of Σ, then there are many different paths from $\mathbf{x}$ to $\mathbf{y}$ through $\Sigma \setminus \{\varnothing\}$. Depending on the path one chooses, the cells of Σ^{+10} visited by the path may have very different values under type_μ. Had we started off with a line segment instead of the triangle τ, things would be simpler: avoiding repetitions there would be a unique path from $\mathbf{x}$ to $\mathbf{y}$. Hence we can prove the following lemma.

Lemma **7.11**. Let $\mathbb{P} \in [\mathrm{Prop}]^{<\aleph_0}$ and $c = \#\mathscr{P}\mathbb{P}$ and $n \geq \log_2(c^4 + 2c + 1)$. Let λ be a line segment, Λ a triangulation of λ and $\mu : \Lambda \to \mathscr{P}\mathbb{P}$ a marking. Then there exists a marking $\underline{\mu} : \mathrm{fac}(\lambda)^{+n} \to \mathscr{P}\mathbb{P}$ such that $\mathrm{type}_{\underline{\mu}}$ and type_μ agree on $\varnothing$ and on the endpoints of λ.

Proof sketch. Up to simplicial isomorphisms, choosing a triangulation of λ merely amounts to choosing the number of vertices. $\mathrm{fac}(\lambda)^{+n}$ has

$$2^n + 1 \geq c^4 + 2c + 2$$

vertices. Suppose that Λ has strictly more than $c^4 + 2c + 2$ vertices. Then Λ has strictly more than $c^4 + 2c + 1$ line segments. Hence, by the pigeonhole principle, there exists a color $\mathbb{C} \subseteq \mathbb{P}$ such that Λ has at least $c^3 + 3$ line segments which are mapped to $(\mathbb{C}, \varnothing)$ by μ. Note that $\#\big(\mathrm{type}_\mu[\Lambda]\big) \leq c^3 + 1$. Hence there exist two distinct line segments $\lambda_0, \lambda_1 \in \Lambda$ such that $\mu(\lambda_0) = (\mathbb{C}, \varnothing) = \mu(\lambda_1)$ and $\mathrm{type}_\mu[\Pi] = \mathrm{type}_\mu[\Lambda]$, where Π is the subcomplex of Λ consisting of all cells that do *not* lie between λ_0 and λ_1 (i.e. $|\Pi| = \lambda \setminus \mathrm{Conv}\big((\mathrm{relInt}\,\lambda_0) \cup (\mathrm{relInt}\,\lambda_1)\big)$). Then we can remove all vertices between λ_0 and λ_1, without changing the types of $\varnothing$ and the endpoints of λ. Repeating this argument, we eventually must have that Λ has at most $c^4 + 2c + 2$ vertices. This proves the lemma. $\square$

Let

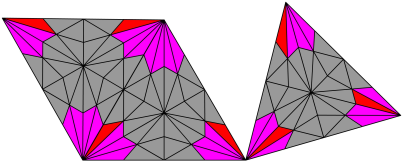


Figure 7.3: sketch of the various $[C,\mathbf{x}]$ (pink, red) and $\tau(\mathbf{x},C)$ (red)

For each $C \in \mathscr{C}(\mathbf{x})$ with $[C,\mathbf{x}] \neq \varnothing$, choose some $\tau(\mathbf{x},C) \in [C,\mathbf{x}]$. See Figure 7.3 for the situation in three triangles of $\Sigma^{+n(1)}$. Let $n(3) = n(2) + \lceil \log_2(c^4 - c + 2) \rceil + 1$. By Lemma 7.8 (and Lemma 2.50-2) there exists a partition $\mathscr{T}(\mathbf{x},C) = \{T_0(\mathbf{x},C), \dots, T_{c^4-c+1}(\mathbf{x},C)\}$ of the set of triangles in $\Sigma^{+n(3)}$ lying in $\tau(\mathbf{x},C)$ such that

- $\#\mathscr{T}(\mathbf{x},C) = c^4 - c + 2$;
- for each $T \in \mathscr{T}(\mathbf{x},C)$ there exists $\tau \in T$ with $\mathbf{x} \in \tau$;
- whenever a triangle in $T_i(\mathbf{x},C)$ and a triangle in $T_j(\mathbf{x},C)$ intersect (other than at $\mathbf{x}$), we have $j \in \{i-1, i, i+1\}$;
- all triangles in $\Sigma^{+n(3)}$ lying in $\tau(\mathbf{x},C)$ and intersecting $\partial\tau(\mathbf{x},C) \setminus \{\mathbf{x}\}$ are in $T_{c^4-c+1}(\mathbf{x},C)$.

Let $\Delta(\mathbf{x},C)$ be the subcomplex of $\Sigma^{+n(3)}$ with carrier

$$\left(\bigcup\left([C,\mathbf{x}] \setminus \{\tau(\mathbf{x},C)\}\right)\right) \cup \left(\bigcup\left(T_{c^4-c+1}(\mathbf{x},C)\right)\right).$$

Let

$$\Lambda_0(\mathbf{x},C) = \Delta(\mathbf{x},C) \cap \downarrow^{\Sigma^{+n(3)}}\left(T_{c^4-c}(\mathbf{x},C)\right).$$

Let $\Lambda_1(\mathbf{x},C)$ be the set of cells of $\Delta(\mathbf{x},C)$ that are not $\mathbf{x}$ and that have a successor (in $\Sigma^{+n(3)}$) which is not contained in $\bigcup[C,\mathbf{x}]$. Then $\Lambda_0(\mathbf{x},C)$ and $\Lambda_1(\mathbf{x},C)$ are subcomplexes of $\Delta(\mathbf{x},C)$. See Figure 7.4. It is easy to check that $\Lambda_0(\mathbf{x},C)$ and $\Lambda_1(\mathbf{x},C)$ are disjoint and each have dimension at most 1:

- Their carriers are disjoint because all successors of cells in $\left(\downarrow^{\Sigma^{+n(3)}}\left(T_{c^4-c}(\mathbf{x},C)\right)\right) \setminus \{\mathbf{x}\}$ are contained in $\bigcup[C,\mathbf{x}]$ since $\left(\bigcup T_{c^4-c}\right) \setminus \{\mathbf{x}\} \subseteq \operatorname{relInt}\tau(\mathbf{x},C)$.

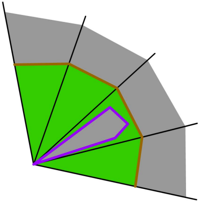


Figure 7.4: sketch of complexes $\Delta(\mathbf{x},C)$ (green), $\Lambda_0(\mathbf{x},C)$ (purple) and $\Lambda_1(\mathbf{x},C)$ (brown)

Let $k=\lceil \log_2(c^2)\rceil$ and $n=n(3)+k$. By Lemma 7.9 there exists a partition $\mathscr{T}'(\mathbf{x},C)=\{T'_0(\mathbf{x},C),\dots,T'_{c^2-1}(\mathbf{x},C)\}$ of the set of triangles in $\Delta(\mathbf{x},C)^{+k}$ such that:

- $\#\mathscr{T}'(\mathbf{x},C)=c^2$;
- whenever a triangle in $T'_i(\mathbf{x},C)$ and a triangle in $T'_j(\mathbf{x},C)$ intersect, we have $j\in\{i-1,i,i+1\}$;
- all triangles in $\Delta(\mathbf{x},C)^{+k}$ that intersect $|\Lambda_0(\mathbf{x},C)|$ are in $T'_0(\mathbf{x},C)$;
- all triangles in $\Delta(\mathbf{x},C)^{+k}$ that intersect $|\Lambda_1(\mathbf{x},C)|$ are in $T'_{c^2-1}(\mathbf{x},C)$.

We have

$$n\le c^3+3\cdot\left(\log_2(c^4+1)+1\right)+4\le c^3+18c^3+4c^3=\overline{n}$$

since $\log_2(c^4+1)+1\le\log_2(c^4)+2\le 4c+2\le 6c^3$.

Let

$$X=\operatorname{vtc}(\Sigma)\sqcup\bigsqcup_{D\in\mathscr{D}}\operatorname{Im} f_D\subseteq\operatorname{vtc}(\Sigma^{+n(1)}).$$

For $\mathbf{x}\in X$ and $C\in\mathscr{C}(\mathbf{x})$, we define a set $\mathfrak{E}(\mathbf{x},C)$ of edge-types as follows. If $\mathbf{x}\in\operatorname{vtc}(\Sigma)$, let

$$\begin{aligned}\mathfrak{E}(\mathbf{x},C)&=\left\{\mathbb{E}_0(\mathbf{x},C),\dots,\mathbb{E}_{c^3-1}(\mathbf{x},C)\right\}\\&=\operatorname{type}_{\mu'}\left[\left\{\lambda'\in\Sigma':\dim\lambda'=1\ \&\ \mathbf{x}\in\lambda'\subseteq\bigcup C\right\}\right].\end{aligned}$$

If $\mathbf{x}=f_D(\mathbb{V})$, let $\mathfrak{E}(\mathbf{x},C)=\mathbb{V}_1$.

Observe that $G\left(\mathfrak{E}(\mathbf{x},C)\right)$ is connected in any case: if $\mathbf{x}=f_D(\mathbb{V})$, this follows from the fact that

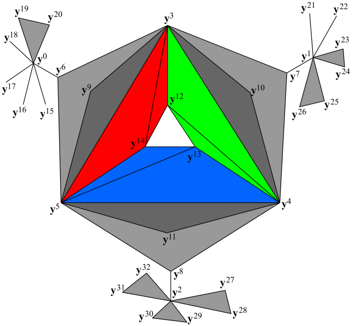


Figure 7.5: the complex Δ

isomorphic to $\Uparrow^{\Theta''}(\mathbf{x})$. Hence by Lemma 5.19, the polyhedron $\left|\operatorname{link}(\Theta'',\mathbf{x})\right|$ has at most two connected components. By Lemma 5.27, it follows that also $\left|\operatorname{link}(\Sigma,\mathbf{x})\right|$ has at most two components. The desired result follows from Lemma 5.19. $\square$

We are now ready for the main result of this section. We shall identify a specific formula, namely $\chi(\Delta)$, and a list $\Sigma_6, \Sigma_9, \Sigma_{12}, \dots$ of simplicial complexes such that

$$\chi(\Delta) \notin \operatorname{Log}_{\varnothing}\left(|\Sigma_k|\right)$$

but, if $n(k)$ is the smallest natural number for which $\chi(\Delta) \notin \operatorname{Log}(\Sigma_k^{+n(k)})$ then

$$\sup\left\{n(6), n(9), n(12), \dots\right\} = \infty.$$

This means that the property expressed by $\chi(\Delta)$ is sufficiently complex that it cannot be translated in terms of some fixed amount of iterations of the barycentric subdivision. Hence one could say that the property expressed by $\chi(\Delta)$ concerns arbitrarily fine triangulations.

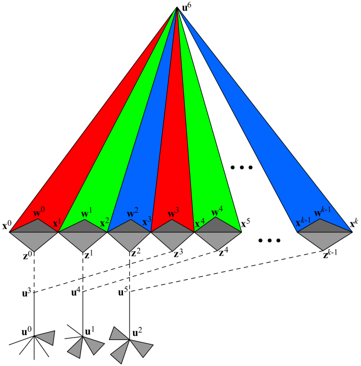


Figure 7.6: the complex Σ_k

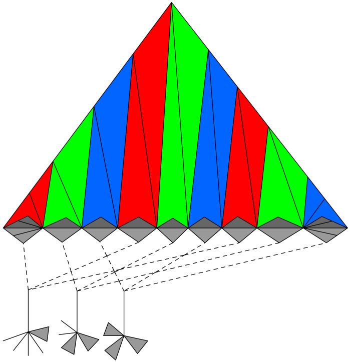


Figure 7.7: suggestion for a p-morphism from a subdivision of Σ_9 to Δ

Dynamic logics of polyhedra and their application in 3D modeling

MSc Thesis *(Afstudeerscriptie)*

written by

Kirill Kopnev
(born 23.11.1999 in Moscow, Russia)

under the supervision of **Nick Bezhanishvili** and **Vincenzo Ciancia**, and submitted to the Examinations Board in partial fulfillment of the requirements for the degree of

MSc in Logic

at the *Universiteit van Amsterdam*.

Date of the public defense:	**Members of the Thesis Committee:**
25.08.2023	dr. Balder ten Cate (chair)
	dr. David Gabelaia
	dr. Alexandru Baltag
	dr. Nick Bezhanishvili (supervisor)
	dr. Vincenzo Ciancia (supervisor)

INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

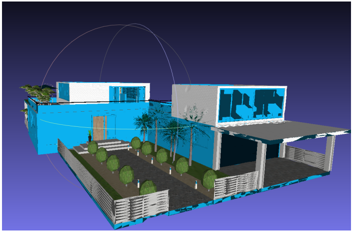


Figure 4.1: The initial 3D model of a building.

3-dimensional simplexes. Along with this data, we also had to specify the propositions we would assign to the relative interiors of simplexes. In our real-world 3D meshes, each simplex is associated with some material (e.g. wood, stone, etc.). The program can be found in the fork of the VoxLogicA at our GitHub repository[4] under the name `program.py`.

Model checking Once the file has been parsed, we can write the text file that will specify the task for our model checker. Figure 4.3 illustrates an example of such a file. The code is self-explanatory, and we will explain only operations `near` and `through`. Operator `near` stands for taking the topological closure of simplexes. We use it to have all the simplexes (i.e. triangles, intervals, and vertices) that satisfy given variables. Operator `trough` stands for γ operator. So, our final variable `house` denotes all the points on the wall or floor such that it is possible to reach the floor from them by passing only through the wall.

In addition to the polyhedra model checker, a visualizer was also presented in [Bez+22]. It takes as input the `.json` file with the loaded model and with the output of `PolyLogicA` and outputs the visualization of the result. In our case, the visualization of the query formulated in 4.3 can be found in Figure 4.4. One can see that the result precisely depicts the areas that we wanted to separate.

[4] [illegible]

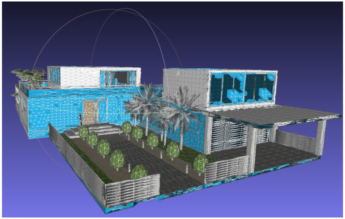


Figure 4.2: Trinagulated model

In summary, the skeleton of the procedure is as follows:

1. Take a 3D model and triangulate it;
2. Check whether the regions that you want to run your query on are connected;
3. Parse the model in order to obtain a `.json` file with simplices and materials for it;
4. Write the `.json` file identifying the region that is intended to be extracted;
5. Inspect the visualization of the result using the visualizer.

4.2.2 Outlook: efficient model checking for dynamic 3D models.

In this subsection, we present how the theory we developed in the previous chapters can be applied to building a prototype for a model checker of dynamic models. The definition of dynamic systems we provided in Chapter 3 underpins a more theoretical view of the dynamics rather than a view of applications. This is because the model $\mathcal{O} = (P, K, R, V)$ is a monolith in which the entire dynamical component of the model is hidden in the relation R. Whereas in reality, when modelling processes, we deal with a discrete set of states, each of which reflects some particular state of the model. From this point of view, real dynamics resembles Snapshot

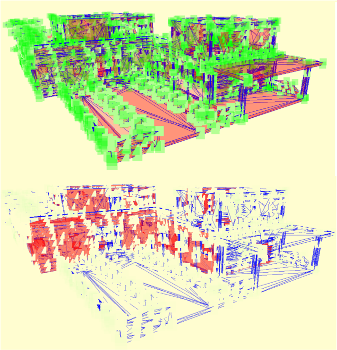


Figure 4.4: The top image visualizes the model of the villa in the visualizer of PolyLogicA, without application of the query from Figure 4.3. The green squares denote 0 - simplexes, blue lines denote 1 - simplexes, and red planes denote 2 - simplexes. As we can see, the villa model contains two blocks of building, an adjoining territory in front of the left block, and a little patio in front of the right block. The query in Figure 4.3 aims to separate the two blocks, excluding the adjoining territory and the patio. The image on the bottom shows a picture after applying the query from Figure 4.3. The red planes denote the result of the query. We hide the 0-simplexes so that they do not distract us from the result. However, visualizer leaves empty space in their places. Overall, we can see that the model checker has extracted exactly two blocks of the building as wanted.

Part 3: Polyhedral reachability logic

Part 3: Polyhedral reachability logic

A: Polyhedral model checking

Polyhedral model checking

Spatial model checking is model checking applied to spatial structures and spatial logic.

Polyhedral model checking

Spatial model checking is model checking applied to spatial structures and spatial logic.

We developed polyhedral model checker to reason about 3D images.

Polyhedral model checking

Spatial model checking is model checking applied to spatial structures and spatial logic.

We developed polyhedral model checker to reason about 3D images.

The key observation is that the poset obtained by a triangulation keeps all the "logical information" about the polyhedra.

Polyhedral model checking

Spatial model checking is model checking applied to spatial structures and spatial logic.

We developed polyhedral model checker to reason about 3D images.

The key observation is that the poset obtained by a triangulation keeps all the “logical information” about the polyhedra.

I’ll show our prototype.

Logical Methods in Computer Science
Volume 18, Issue 4, 2022, pp. 7:1–7:38
https://lmcs.episciences.org/

Submitted Feb. 08, 2022
Published Nov. 22, 2022

GEOMETRIC MODEL CHECKING OF CONTINUOUS SPACE

NICK BEZHANISHVILI [a], VINCENZO CIANCIA [b], DAVID GABELAIA [c], GIANLUCA GRILLETTI [d], DIEGO LATELLA [b], AND MIEKE MASSINK [b]

[a] Institute for Logic, Language and Computation, University of Amsterdam, The Netherlands
e-mail address: n.bezhanishvili@uva.nl

[b] Istituto di Scienza e Tecnologie dell'Informazione "A. Faedo", Consiglio Nazionale delle Ricerche, Pisa, Italy
e-mail address: {vincenzo.ciancia,diego.latella,mieke.massink}@cnr.it

[c] TSU Razmadze Mathematical Institute, Tbilisi, Georgia
e-mail address: gabelaia@gmail.com

[d] Munich Center for Mathematical Philosophy, Ludwig-Maximilians-Universität München, Germany
e-mail address: grilletti.gianluca@gmail.com

Abstract. Topological Spatial Model Checking is a recent paradigm where model checking techniques are developed for the topological interpretation of Modal Logic. The Spatial Logic of Closure Spaces, SLCS, extends Modal Logic with reachability connectives that, in turn, can be used for expressing interesting spatial properties, such as "being near to" or "being surrounded by". SLCS constitutes the kernel of a solid logical framework for reasoning about *discrete* space, such as graphs and digital images, interpreted as quasi discrete closure spaces. Following a recently developed *geometric* semantics of Modal Logic, we propose an interpretation of SLCS in *continuous space*, admitting a geometric spatial model checking procedure, by resorting to models based on polyhedra. Such representations of space are increasingly relevant in many domains of application, due to recent developments of 3D scanning and visualisation techniques that exploit mesh processing. We introduce PolyLogicA, a geometric spatial model checker for SLCS formulas on polyhedra and demonstrate feasibility of our approach on two 3D polyhedral models of realistic size. Finally, we introduce a geometric definition of bisimilarity, proving that it characterises logical equivalence.

Geometric model checking

We assume that each polyhedron P comes with a fixed triangulation and its poset Σ.

Geometric model checking

We assume that each polyhedron P comes with a fixed triangulation and its poset Σ.

Another term: simplicial complex.

Geometric model checking

We assume that each polyhedron P comes with a fixed triangulation and its poset Σ.

Another term: simplicial complex.

We add the reachability modality $\gamma(\varphi, \psi)$ to our language.

Geometric model checking

We assume that each polyhedron P comes with a fixed triangulation and its poset Σ.

Another term: simplicial complex.

We add the reachability modality $\gamma(\varphi, \psi)$ to our language.

This is a variant of a spatial Until operation.

Geometric model checking

We assume that each polyhedron P comes with a fixed triangulation and its poset Σ.

Another term: simplicial complex.

We add the reachability modality $\gamma(\varphi, \psi)$ to our language.

This is a variant of a spatial Until operation.

$\mathcal{M}, x \models \gamma(\varphi, \psi) \Leftrightarrow$ there exists a path $\pi : [0, 1] \to P$ such that $\pi(0) = x$, $\pi(1) \in [\![\psi]\!]$ and $\pi[(0, 1)] \subseteq [\![\varphi]\!]$.

Up-down paths

Definition. Let $\mathcal{M} = (W, \preccurlyeq, V)$ be a poset model.

Up-down paths

Definition. Let $\mathcal{M} = (W, \preccurlyeq, V)$ be a poset model.

A sequence $(w_0, \ldots, w_k) \subseteq W$ is said to be an up-down path if $k = 2j$ for some $j > 0$, $w_0 \preccurlyeq w_1$, $w_{k-1} \succcurlyeq w_k$, and whenever $0 < i < j$, we have that $w_{2i-1} \succ w_{2i} \prec w_{2i+1}$.

Up-down paths

Definition. Let $\mathcal{M} = (W, \preccurlyeq, V)$ be a poset model.

A sequence $(w_0, \ldots, w_k) \subseteq W$ is said to be an up-down path if $k = 2j$ for some $j > 0$, $w_0 \preccurlyeq w_1$, $w_{k-1} \succcurlyeq w_k$, and whenever $0 < i < j$, we have that $w_{2i-1} \succ w_{2i} \prec w_{2i+1}$.

Thus, an up-down path is a path
$w_0 \preccurlyeq w_1 \succ w_2 \prec w_3 \succ \ldots \prec w_{k-1} \succcurlyeq w_k$.

Up-down paths

Definition. Let $\mathcal{M} = (W, \preccurlyeq, V)$ be a poset model.

Up-down paths

Definition. Let $\mathcal{M} = (W, \preccurlyeq, V)$ be a poset model.

$\mathcal{M}, x \models \gamma(\varphi, \psi)$ iff there is an up-down path $(w_0, \ldots, w_k) \subseteq W$ such that $w_0 = x$, $\mathcal{M}, w_k \models \psi$ and $\mathcal{M}, w_i \models \varphi$ for $0 < i < k$.

Up-down paths

Definition. Let $\mathcal{M} = (W, \preccurlyeq, V)$ be a poset model.

$\mathcal{M}, x \models \gamma(\varphi, \psi)$ iff there is an up-down path $(w_0, \ldots, w_k) \subseteq W$ such that $w_0 = x$, $\mathcal{M}, w_k \models \psi$ and $\mathcal{M}, w_i \models \varphi$ for $0 < i < k$.

$$\mathcal{M}, x \models \Diamond\varphi \text{ iff } \mathcal{M}, x \models \gamma(\varphi, \top).$$

Up-down paths

Definition. Let $\mathcal{M} = (W, \preccurlyeq, V)$ be a poset model.

$\mathcal{M}, x \models \gamma(\varphi, \psi)$ iff there is an up-down path $(w_0, \ldots, w_k) \subseteq W$ such that $w_0 = x$, $\mathcal{M}, w_k \models \psi$ and $\mathcal{M}, w_i \models \varphi$ for $0 < i < k$.

$$\mathcal{M}, x \models \Diamond\varphi \text{ iff } \mathcal{M}, x \models \gamma(\varphi, \top).$$

Theorem. A polyhedron P satisfies φ iff its poset Σ satisfies φ.

Up-down paths

Definition. Let $\mathcal{M} = (W, \preccurlyeq, V)$ be a poset model.

$\mathcal{M}, x \models \gamma(\varphi, \psi)$ iff there is an up-down path $(w_0, \ldots, w_k) \subseteq W$ such that $w_0 = x$, $\mathcal{M}, w_k \models \psi$ and $\mathcal{M}, w_i \models \varphi$ for $0 < i < k$.

$$\mathcal{M}, x \models \Diamond\varphi \text{ iff } \mathcal{M}, x \models \gamma(\varphi, \top).$$

Theorem. A polyhedron P satisfies φ iff its poset Σ satisfies φ.

Weak Simplicial Bisimilarity for Polyhedral Models and $\mathtt{SLCS}_\eta$*

Nick Bezhanishvili[1] [0009−0005−6692−5051], Vincenzo Ciancia[2] [0000−0003−1314−0574], David Gabelaia[3] [0000−0002−8317−7949], Mamuka Jibladze[3] [0000−0002−9434−9523], Diego Latella[2] [0000−0002−3257−9059], Mieke Massink[2] [0000−0001−5089−002X], and Erik P. de Vink[4] [0000−0001−9514−2260]

[1] Institute for Logic, Language and Computation, University of Amsterdam, The Netherlands `n.bezhanishvili@uva.nl`

[2] Istituto di Scienza e Tecnologie dell'Informazione "A. Faedo", Consiglio Nazionale delle Ricerche, Pisa, Italy
`{Vincenzo.Ciancia, Diego.Latella, Mieke.Massink}@cnr.it`

[3] Andrea Razmadze Mathematical Institute, I. Javakhishvili Tbilisi State University, Georgia `{gabelaia, mamuka jibladze}@gmail.com`

[4] Eindhoven University of Technology, The Netherlands `evink@win.tue.nl`

Abstract. In the context of spatial logics and spatial model checking for polyhedral models — mathematical basis for visualisations in continuous space — we propose a weakening of simplicial bisimilarity. We additionally propose a corresponding weak notion of ±-bisimilarity on cell-poset models, discrete representation of polyhedral models. We show that two points are weakly simplicial bisimilar iff their representations are weakly ±-bisimilar. The advantage of this weaker notion is that it leads to a stronger reduction of models than its counterpart that was introduced in our previous work. This is important, since real-world polyhedral models, such as those found in domains exploiting mesh processing, typically consist of large numbers of cells. We also propose $\mathtt{SLCS}_\eta$, a weaker version of the *Spatial Logic for Closure Spaces* (SLCS) on polyhedral models, and we show that the proposed bisimilarities enjoy the Hennessy-Milner property: two points are weakly simplicial bisimilar iff they are logically equivalent for $\mathtt{SLCS}_\eta$. Similarly, two cells are weakly ±-bisimilar iff they are logically equivalent in the poset-model interpretation of $\mathtt{SLCS}_\eta$. This work is performed in the context of the geometric spatial model checker

Up-down bisimulations

Definition. By picture.

Up-down bisimulations

Definition. By picture.

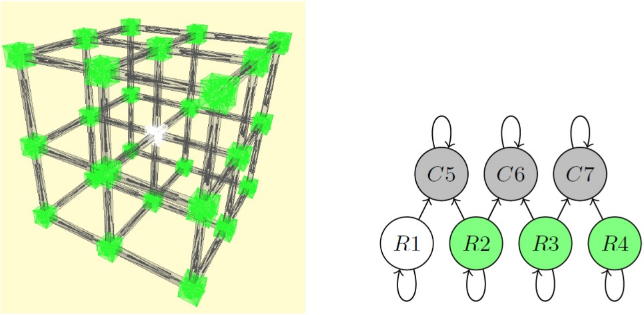
Theorem. Up-down bisimilar worlds satisfy the same formulas.

Up-down bisimulations

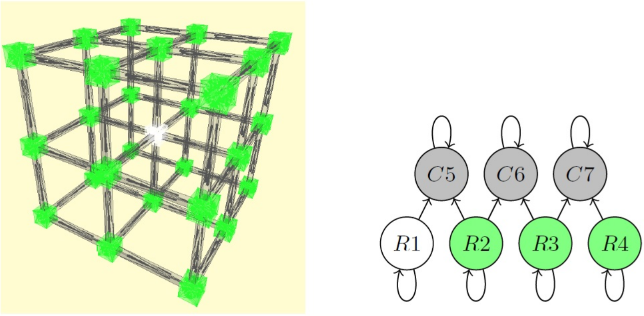
Definition. By picture.

Theorem. Up-down bisimilar worlds satisfy the same formulas.

We will use bisimulation quotients.

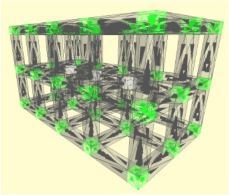


(a) (b)

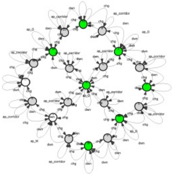


(a) (b)

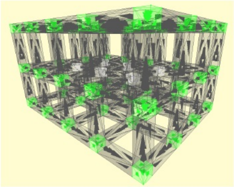
11,205 cells get reduced to just 7 in the minimal model.



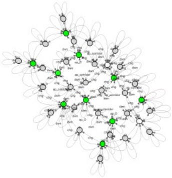
(a) Cube 3x5x3



(b) Minimised LTS

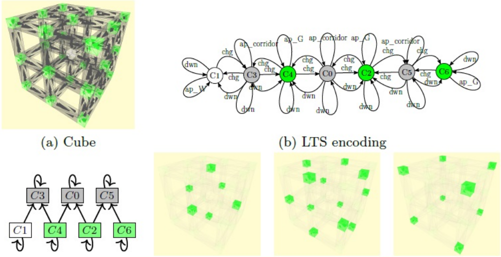


(c) Cube 3x5x4



(d) Minimised LTS

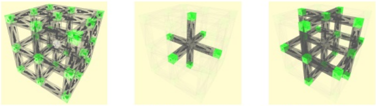
Fig. 11: Cubes of dimension 3x5x3 (Fig. 11a) and 3x5x4 (Fig. 11c) and their respective minimal LTSs (Figs. 11b and 11d).



(a) Cube (b) LTS encoding

(c) Minimal model (d) C4 (e) C2 (f) C6

Fig. 6: Cube with 27 rooms: 26 green and one white in the middle.



(a) (b) Result ϕ_1 (c) Result ϕ_2

Fig. 7: (7a) The 3D cube. Results of `PolyLogicA` model checking of the formulas ϕ_1 (7b) and ϕ_2 (7c) on the minimised model mapped back onto the full 3D cube with `PolyVisualizer`.

Part 3: Polyhedral reachability logic

Part 3: Polyhedral reachability logic

B: Axiomatization and completeness

Logics of polyhedral reachability

Nick Bezhanishvili

University of Amsterdam, Amsterdam, The Netherlands

Laura Bussi Vincenzo Ciancia

National Research Council, Pisa, Italy

David Fernández-Duque

University of Barcelona, Barcelona, Spain

David Gabelaia

TSU Razmadze Mathematical Institute, Tbilisi, Georgia

Abstract

Polyhedral semantics is a recently introduced branch of spatial modal logic, in which modal formulas are interpreted as piecewise linear subsets of an Euclidean space. Polyhedral semantics for the basic modal language has already been well investigated. However, for many practical applications of polyhedral semantics, it is advantageous to enrich the basic modal language with a reachability modality. Recently, a language with an Until-like spatial modality has been introduced, with demonstrated applicability to the analysis of 3D meshes via model checking. In this paper, we exhibit an axiom system for this logic, and show that it is complete with respect to polyhedral semantics. The proof consists of two major steps: First, we show that this logic,

Polyhedral reachability logic

Axioms of the Alexandroff reachability logic ALR are given by all the propositional tautologies and Modus Ponens, S4 axioms and rules for $\Box$, plus the following:

Axiom 1. $\psi \vee (\varphi \wedge \gamma(\varphi, \psi)) \to \Box(\varphi \to \gamma(\varphi, \psi))$

Axiom 2. $\Diamond(\varphi \wedge \gamma(\varphi, \psi)) \to \gamma(\varphi, \psi)$

Rule 1. $$\frac{\varphi \to \varphi' \quad \psi \to \psi'}{\gamma(\varphi, \psi) \to \gamma(\varphi', \psi')}$$

Rule 2. $$\frac{\psi \to \Box(\varphi \to \psi) \quad \varphi \wedge \Diamond(\varphi \wedge \psi) \to \psi}{\gamma(\varphi, \psi) \to \Diamond(\varphi \wedge \psi)}.$$

The polyhedral reachability logic PLR is obtained by adding the Grz axiom $\Box(\Box(p \to \Box p) \to p) \to \Box p$ to ALR.

Soundness

Proof by Picture.

Completeness

Proof by Filtration.

Completeness

Proof by Filtration.

For ALR we use transitive filtration.

Completeness

Proof by Filtration.

For ALR we use transitive filtration.

For PLR we use transitive filtration and then cut clusters.

Completeness

Proof by Filtration.

For ALR we use transitive filtration.

For PLR we use transitive filtration and then cut clusters.

Theorem. ALR and PLR have the finite model property (are complete for finite models).

Polyhedral completeness

Suppose φ is PLR consistent.

Polyhedral completeness

Suppose φ is PLR consistent.

Then there is a finite poset model M such that φ is satisfied at M.

Polyhedral completeness

Suppose φ is PLR consistent.

Then there is a finite poset model M such that φ is satisfied at M.

Take the nerve $\mathcal{N}(M)$ of M.

Polyhedral completeness

Suppose φ is PLR consistent.

Then there is a finite poset model M such that φ is satisfied at M.

Take the nerve $\mathcal{N}(M)$ of M.

Lemma: If φ is satisfied at M, it is satisfied at $\mathcal{N}(M)$.

Polyhedral completeness

Suppose φ is PLR consistent.

Then there is a finite poset model M such that φ is satisfied at M.

Take the nerve $\mathcal{N}(M)$ of M.

Lemma: If φ is satisfied at M, it is satisfied at $\mathcal{N}(M)$.

Now we construct a polyhedron P from $\mathcal{N}(M)$ such that
P satisfies ψ iff $\mathcal{N}(M)$ satisfies ψ.

Polyhedral completeness

Suppose φ is PLR consistent.

Then there is a finite poset model M such that φ is satisfied at M.

Take the nerve $\mathcal{N}(M)$ of M.

Lemma: If φ is satisfied at M, it is satisfied at $\mathcal{N}(M)$.

Now we construct a polyhedron P from $\mathcal{N}(M)$ such that
P satisfies ψ iff $\mathcal{N}(M)$ satisfies ψ.

So φ is satisfied on a polyhedron.

Polyhedral completeness

Suppose φ is PLR consistent.

Then there is a finite poset model M such that φ is satisfied at M.

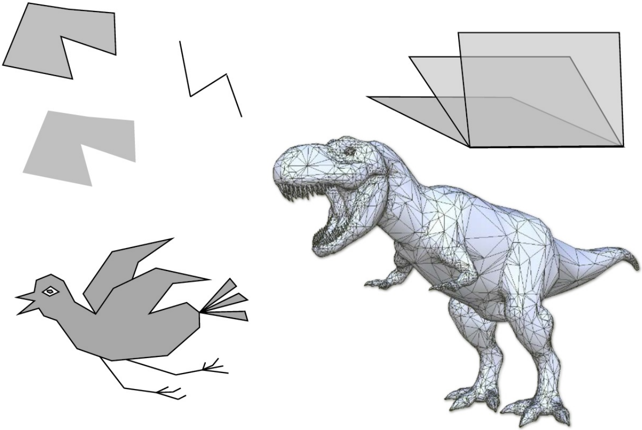
Take the nerve $\mathcal{N}(M)$ of M.

Lemma: If φ is satisfied at M, it is satisfied at $\mathcal{N}(M)$.

Now we construct a polyhedron P from $\mathcal{N}(M)$ such that
P satisfies ψ iff $\mathcal{N}(M)$ satisfies ψ.

So φ is satisfied on a polyhedron.

Theorem. PLR is polyhedrally complete.



Thank you!