Metric Spaces

Project Topology in and via Logic 2026

Navid Kianfar, Jongsung Song, Idske Roest

UvA – ILLC

January 29th 2026

Table of Contents

Def. Continuity on $\mathbb{R}$

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is *continuous at* $a \in \mathbb{R}$ if

$$\forall \varepsilon > 0\ \exists \delta > 0 \text{ s.t. } |f(x) - f(a)| < \varepsilon \text{ whenever } |x - a| < \delta.$$

Geometric intuition

If x is close to a, then $f(x)$ is close to $f(a)$.

Continuity of $f : \mathbb{R}^2 \to \mathbb{R}$

Definition

Let $f : \mathbb{R}^2 \to \mathbb{R}$ and let $(a, b) \in \mathbb{R}^2$. We say that f is *continuous at* (a, b) if

$$\forall \varepsilon > 0 \; \exists \delta > 0 \text{ s.t. } |f(x, y) - f(a, b)| < \varepsilon$$

$$\text{whenever} \quad \sqrt{(x-a)^2 + (y-b)^2} < \delta.$$

Euclidean distance in $\mathbb{R}^2$

$$d_2\big((x, y), (a, b)\big) = \sqrt{(x-a)^2 + (y-b)^2}.$$

Def. Continuity of $f : \mathbb{R}^n \to \mathbb{R}$

Definition

Let $f : \mathbb{R}^n \to \mathbb{R}$ and let $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$. We say that f is *continuous at a* if

$$\forall \varepsilon > 0 \; \exists \delta > 0 \text{ s.t. } |f(x) - f(a)| < \varepsilon \text{ whenever } \|x - a\|_2 < \delta,$$

where for $x = (x_1, \ldots, x_n)$,

$$\|x - a\|_2 = \left(\sum_{i=1}^{n} (x_i - a_i)^2 \right)^{1/2}$$

Idea

Control the output error $|f(x) - f(a)|$ by making x close to a in Euclidean distance.

Def. Metric and Metric Space

Definition

Let $X \neq \varnothing$. A function $d : X \times X \to \mathbb{R}$ is a *metric* if for all $x, y, z \in X$:

(M1) $d(x, y) \geq 0$, and $d(x, y) = 0 \iff x = y$.

(M2) $d(x, y) = d(y, x)$.

(M3) $d(x, z) \leq d(x, y) + d(y, z)$.

A *metric space* is a pair (X, d).

Interpretation

(M1) no negative distances; (M2) symmetry; (M3) triangle inequality.

Def. Continuity between Metric Spaces

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$. We say f is *continuous at* $x_0 \in X$ if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } d_Y\big(f(x), f(x_0)\big) < \varepsilon \text{ whenever } d_X(x, x_0) < \delta.$$

f is *continuous* if it is continuous at every $x_0 \in X$.

Idea

Small changes in input (in d_X) force small changes in output (in d_Y).

Example: The Euclidean Metric on $\mathbb{R}^n$

Example

For $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, define

$$d_2(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}.$$

Then d_2 is a metric on $\mathbb{R}^n$.

Proof.

- **(M1)** $d_2(x, y) \geq 0$ and $d_2(x, y) = 0$ forces each $x_i = y_i$, hence $x = y$.
- **(M2)** Symmetry: $(x_i - y_i)^2 = (y_i - x_i)^2$.
- **(M3)** Triangle inequality follows from Cauchy–Schwarz: (Continued on next page)

Example: The Euclidean Metric on $\mathbb{R}^n$

continue Proof: (M3).

To be shown: for all $x, y, z \in \mathbb{R}^n$:

$$\left(\sum_{i=1}^{n}(x_i - z_i)^2\right)^{1/2} \leq \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2} + \left(\sum_{i=1}^{n}(y_i - z_i)^2\right)^{1/2}$$

$$\Leftrightarrow \sum_{i=1}^{n}(x_i - z_i)^2 \leq \left(\left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2} + \left(\sum_{i=1}^{n}(y_i - z_i)^2\right)^{1/2}\right)^2$$

$= \sum_{i=1}^{n}(x_i - y_i)^2 + \sum_{i=1}^{n}(y_i - z_i)^2 +$
$2\left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2}\left(\sum_{i=1}^{n}(y_i - z_i)^2\right)^{1/2}$
(Continued on next page)

Example: The Euclidean Metric on $\mathbb{R}^n$

continue Proof: (M3).

Now use $x_i - z_i = (x_i - y_i) + (y_i - z_i)$ and expand:

$$\begin{aligned}\sum_{i=1}^{n}(x_i - z_i)^2 &= \sum_{i=1}^{n}\big((x_i - y_i) + (y_i - z_i)\big)^2 \\ &= \sum_{i=1}^{n}(x_i - y_i)^2 + \sum_{i=1}^{n}(y_i - z_i)^2 + 2\sum_{i=1}^{n}(x_i - y_i)(y_i - z_i)\end{aligned}$$

So it remains to show

$$\sum_{i=1}^{n}(x_i - y_i)(y_i - z_i) \leq \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2}\left(\sum_{i=1}^{n}(y_i - z_i)^2\right)^{1/2},$$

which holds by the Cauchy–Schwarz–ineq (square both sides!). □

Def. Open Balls

Definition

In a metric space (X, d), the *open ball* of radius $r > 0$ centered at $x_0 \in X$ is

$$B_r(x_0) = \{x \in X : \; d(x, x_0) < r\}.$$

Example

(a) In $(\mathbb{R}, |\cdot|)$:

$$B_r(x_0) = (x_0 - r, \; x_0 + r).$$

(b) In $(\mathbb{R}^2, d_2)$: $B_r(x_0)$ is an open disc of radius r.

(c) In $(\mathbb{R}^3, d_2)$: $B_r(x_0)$ is an open ball of radius r.

Continuity via Open Balls

Proposition

Let $f : (X, d_X) \to (Y, d_Y)$ and fix $x_0 \in X$. Then f is continuous at x_0 iff

$$\forall \varepsilon > 0 \; \exists \delta > 0 \text{ s.t. } f\big(B_\delta^{d_X}(x_0)\big) \subseteq B_\varepsilon^{d_Y}\big(f(x_0)\big).$$

Same content as ε–δ

"Points δ-close to x_0 map to points ε-close to $f(x_0)$."

Def. Open Sets in a Metric Space

Definition

Let (X, d) be a metric space and $U \subseteq X$. We call U *open* (or *d-open*) if

$$\forall x \in U \; \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon^d(x) \subseteq U.$$

Example

Any open interval (a, b) in $\mathbb{R}$ is open in $\mathbb{R}$,
but not $[a, b], [a, b), (a, b]$

Continuity and Preimages of Open Sets

Proposition

Let $f : (X, d_X) \to (Y, d_Y)$. Then f is continuous iff for every open set $V \subseteq Y$, the preimage

$$f^{-1}(V) = \{x \in X : \ f(x) \in V\}$$

is open in X.

Table of Contents

Metric to Topological

- Any metric space induces a topological space.

Proposition

Given a metric space (X, d), we have (X, τ_d) is a topological space, where for any $U \subseteq X$

$$U \in \tau_d \quad \textit{iff} \quad U \textit{ is } d\textit{-open in } X.$$

Recall: definition of *d*-open

For every $x \in U$, there exists $\epsilon > 0$ such that $B_\epsilon^d(x) \subseteq U$.

Is (X, τ_d) actually a topological space? Let's check!

- O2 Closure under arbitrary union:
  Let $(U_i)_{i \in I} \subseteq \tau_d$, and $x \in \bigcup_{i \in I} U_i$ arbitrary. Then $x \in U_j$ for some $j \in I$. Since U_j is open by assumption, there exists $\epsilon > 0$ s.t. $B_\epsilon(x) \subseteq U_j$. This implies $B_\epsilon(x) \subseteq \bigcup_{i \in I} U_i$. Hence, $\bigcup_{i \in I} U_i \in \tau_d$ as desired.

Metrizability

Definition (Metrizable spaces)

A topological space arising from a metric space in this way is called *metrizable*.

Example

- The discrete topology is metrizable.
- The Euclidean topology is metrizable

What Makes a Space Metrizable?

How can we tell that a space is metrizable?

Proposition

Any metrizable space is Hausdorff.

Proof.

Let (X, d) be a metric space, and consider τ_d. If $x, y \in X$ with $x \neq y$, then $d(x, y) > 0$. Consider $\epsilon := \frac{d(x,y)}{2}$ Then, the open balls $B_\epsilon(x), B_\epsilon(y)$ are disjoint open sets containing x and y respectively. □

Metrization Theorems

But what about the other direction?

Theorem (Urysohn's metrization theorem)

Every topological space that is normal and second-countable is metrizable.

Normal and Second-countable

Recall

X normal if whenever E, F disjoint closed subsets of X, then there exist open sets U, V with $E \subseteq U$ and $F \subseteq V$ such that $U \cap V = \emptyset$.

Definition (Second-countable)

A topological space is second-countable if it has a countable base.

Proof Sketch Urysohn's Metrization Theorem

Lemma (Urysohn's lemma)

A topological space (X, τ) is normal if, and only if, for every pair of nonempty, closed, disjoint subsets $E, F \subseteq X$, there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ for all $x \in E$, and $f(x) = 1$ for all $x \in F$.

Proof sketch Urysohn's Metrization Theorem

Idea:

- Apply Urysohn's lemma countably many times.
- We get a *countable* family of continuous functions $\{f_n : X \to [0,1]\}_{n\in\mathbb{N}}$.
- For any $x \in X$ and neighbourhood U of x, some f_m is strictly positive at x, and 0 outside U $\leftarrow$ second-countability.
- Define $F : X \to \mathbb{R}^{\mathbb{N}}$, $x \mapsto (f_1(x), f_2(x), \dots)$. Can be checked that this is a homeomorphism into its image $F(X)$.
- $\mathbb{R}^{\mathbb{N}}$ with the product topology is metrizable.

Example: Non-metrizable Spaces

- $\{0, 1\}$ with indiscrete topology is not Hausdorff.
- Sorgenfrey line is not second-countable.

Table of Contents

Sequence & Convergence

Definition (Sequence)

Given a metric space (X, d), a sequence on this space is a map $s : \mathbb{N} \to X$ commonly denoted as (x_n), having $x_n = s(n)$.

Definition (Convergence)

A sequence (x_n) converges to a point x, if for any $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for any $n \geq N$.

Sequence & Convergence (Cont.)

Example

- The sequence (x_n) defined as $x_n = \frac{1}{n},\ n \in \mathbb{N}$ converges to the limit point 0.
- Consider some non-empty metric space containing at least 2 points x and y. The sequence $x_n = \begin{cases} x, & \text{if n is even,} \\ y, & \text{else.} \end{cases}$ does not converge.

- Note: A converging sequence converges to a unique point.

Cauchy Sequences

Definition (Cauchy Sequence)

A sequence defined on a metric space (X, d) is said to be Cauchy, if for any $\epsilon > 0$ there is some $N \in \mathbb{N}$ such that for all $n, m > N$ we have $d(x_n, x_m) < \epsilon$.

Cauchy Sequences

Definition (Cauchy Sequence)

A sequence defined on a metric space (X, d) is said to be Cauchy, if for any $\epsilon > 0$ there is some $N \in \mathbb{N}$ such that for all $n, m > N$ we have $d(x_n, x_m) < \epsilon$.

Proposition

All convergent sequences are Cauchy.

Metric Definitions Using Convergence

Suppose (X, d) is a metric space.

Definition (Limit point)

A point x in a space X is called a limit point of a subset $A \subseteq X$, if given any $\epsilon > 0$ we have $B_\epsilon \setminus \{x\} \cap A \neq \emptyset$. In other words, there is some close neighbor other than x itself that intersects A.

Metric Definitions Using Convergence

Suppose (X, d) is a metric space.

Definition (Limit point)

A point x in a space X is called a limit point of a subset $A \subseteq X$, if given any $\epsilon > 0$ we have $B_\epsilon \setminus \{x\} \cap A \neq \emptyset$. In other words, there is some close neighbor other than x itself that intersects A.

Proposition (Limit point (alt. def.))

A point x in a space X is deemed a limit point of a subset $A \subseteq X$, if there is a sequence (x_n) having $\{x_n\} \subseteq A \setminus \{x\}$ which converges to x.

Limit points formalize the notion of a point being infinitely close to a set without actually being a part of it.

Metric Definitions Using Convergence (Cont.)

Definition (Closed set)

A set A is closed if for any $x \in X$ and $\epsilon > 0$ we have $B_\epsilon(x) \cap A \neq \emptyset$, then $x \in A$.

Metric Definitions Using Convergence (Cont.)

Definition (Closed set)

A set A is closed if for any $x \in X$ and $\epsilon > 0$ we have $B_\epsilon(x) \cap A \neq \emptyset$, then $x \in A$.

Proposition (Closed set (alt. def.))

A set A is closed if for any $(x_n) \subseteq A$ converging to x, we have $x \in A$.

I.e., a set is closed iff it contains all of its limit points.

Metric Definitions Using Convergence (Cont.)

Assume we have two metric spaces (X, d_X) and (Y, d_Y), and a map $f : X \to Y$. The statements below are equivalent.

Proposition

- *The function f is continuous.*
- *For every $x \in X$, whenever $B_\epsilon(f(x)) \subseteq Y$, we have some $f(B_\delta(x)) \subseteq B_\epsilon(f(x))$.*
- *If a sequence $(x_n) \subseteq X$ converges to x; then the sequence $(f(x_n))$ converges to $f(x)$.*
- *for every $S \subseteq X$, $f(\overline{S}) \subseteq \overline{f(S)}$.*

Informally, in context of metric spaces, continuity means respecting distance, which can alternatively be put as respecting the correspondence between limit points.

Completeness

You might have encountered the fact that the real line has *no gaps*. What does this precisely mean?

Definition (Completeness)

Let (X, d) be a metric space. The set X is said to be complete with respect to the metric d, if every Cauchy sequence in X, converges to a point in X.

Completeness

You might have encountered the fact that the real line has *no gaps*. What does this precisely mean?

Definition (Completeness)

Let (X, d) be a metric space. The set X is said to be complete with respect to the metric d, if every Cauchy sequence in X, converges to a point in X.

Example

- The set $(0, \infty)$ equipped with $d(x, y) = |x - y|$ is not complete. Why?

Completeness

You might have encountered the fact that the real line has *no gaps*. What does this precisely mean?

Definition (Completeness)

Let (X, d) be a metric space. The set X is said to be complete with respect to the metric d, if every Cauchy sequence in X, converges to a point in X.

Example

- The set $(0, \infty)$ equipped with $d(x, y) = |x - y|$ is not complete. Why?
  The aforementioned sequence $(x_n) = (1/n)$.

Completeness

You might have encountered the fact that the real line has *no gaps*. What does this precisely mean?

Definition (Completeness)

Let (X, d) be a metric space. The set X is said to be complete with respect to the metric d, if every Cauchy sequence in X, converges to a point in X.

Example

- The set $(0, \infty)$ equipped with $d(x, y) = |x - y|$ is not complete. Why?
  The aforementioned sequence $(x_n) = (1/n)$.
- The same set equipped with the discrete metric is complete.

Completeness (Cont.)

Completeness portrays a notion of gapless-ness as it contains all the points that are infinitely close to it—analogous to how a closed set (or the closure of a set) contains all of its limit points.

Completeness in Topology

- We have now seen that fundamental notions of metric spaces can be abstracted to topology.

Completeness in Topology

- We have now seen that fundamental notions of metric spaces can be abstracted to topology.
- How about completeness?

Completeness in Topology

- We have now seen that fundamental notions of metric spaces can be abstracted to topology.
- How about completeness? The answer is **no**.

Completeness in Topology

- We have now seen that fundamental notions of metric spaces can be abstracted to topology.
- How about completeness? The answer is **no**.
- Take the space $\mathbb{R}$ with the euclidean topology, and its induced subspace $(0, 1)$. We know from the lectures that $\mathbb{R} \cong (0, 1)$.

Completeness in Topology

- We have now seen that fundamental notions of metric spaces can be abstracted to topology.
- How about completeness? The answer is **no**.
- Take the space $\mathbb{R}$ with the euclidean topology, and its induced subspace $(0, 1)$. We know from the lectures that $\mathbb{R} \cong (0, 1)$.
- But one is complete and the other isn't.

References

- Wilson Alexander Sutherland, *Introduction to Metric and Topological Spaces*, Oxford science publications, 1975.
- Robert André, *Point-Set Topology with Topics: Basic General Topology for Graduate Studies*, World Scientific Publishing, 2023.

Thank you for your time!