Topological Categories

Plan

- Motivations
- Definitions
- Examples
- Applications

Motivations

What do Top, Kr, Nb all have in common?

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1. They all have some "nice" properties.

2. They can all serve as semantics for ML.

Plan

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① They all have some "nice" properties.

② They can all serve as semantics for ML.

Top has some "nice" properties:

$(\mathbf{Top},\ U : \mathbf{Top} \to \mathbf{Set})$

forgetful functor

(weak) coarse — indiscrete $\to \tau_\nabla = \{\emptyset, X\}$

$\vdots$

(strong) fine — discrete $\to \tau_\Delta = \mathcal{P}(X)$

complete lattice!

X ← the underlying set

$\Delta \dashv U \dashv \nabla$

product as in Set

the coarsest to make π_1 & π_2 continuous

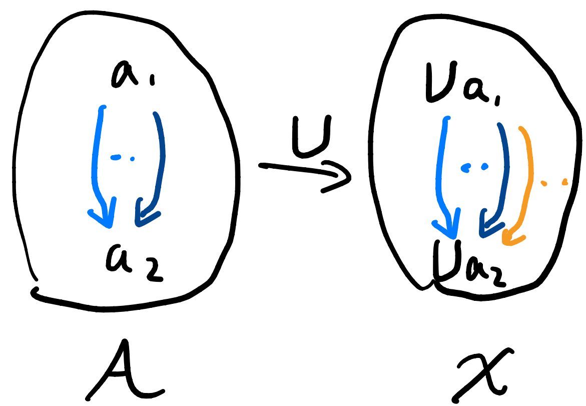
$$\begin{tikzcd} & (X_1 \times X_2, \tau') \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ (X_1, \tau_1) & & (X_2, \tau_2) \end{tikzcd}$$

Top, Kr, Nb all have similar nice properties and let's characterize them.

Definitions

Def. (concrete category & construct)

Let $\mathcal{X}$ be a category. A concrete category over $\mathcal{X}$ is a pair $(\mathcal{A}, U)$, where $\mathcal{A}$ is a category and $U: \mathcal{A} \to \mathcal{X}$ is a faithful functor.



the operation

$$\mathrm{Hom}_{\mathcal{A}}(a_1, a_2) \to \mathrm{Hom}_{\mathcal{X}}(Ua_1, Ua_2)$$
$$f \mapsto Uf$$

is injective.

base category

(when $\mathcal{X}$ is Set, $(\mathcal{A}, U)$ is called construct)

Examples:

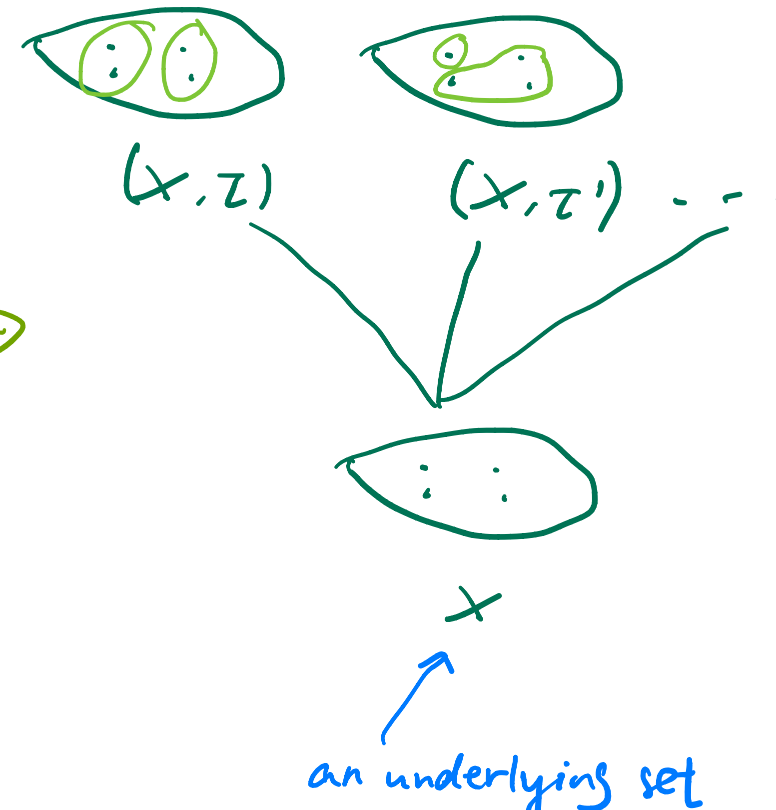
Examples:

$$\mathcal{C} \xrightarrow{id_{\mathcal{C}}} \mathcal{C}$$

$$\mathbf{Top} \xrightarrow{U} \mathbf{Set}$$

$$\mathbf{Gr} \xrightarrow{U} \mathbf{Set}$$

$$\vdots$$



Def. (fibre)

Let $(\mathcal{A}, U)$ be a concrete category over $\mathcal{X}$.

The fibre of an $\mathcal{X}$-object x is the preordered class consisting of all $\mathcal{A}$-objects A with $U(A) = X$ ordered by:

$$A \leq B \text{ iff } id_X : UA \to UB \text{ is an } \mathcal{A}\text{-morphism.}$$

Examples:

$(\mathbf{Top}, U : \mathbf{Top} \to \mathbf{Set})$

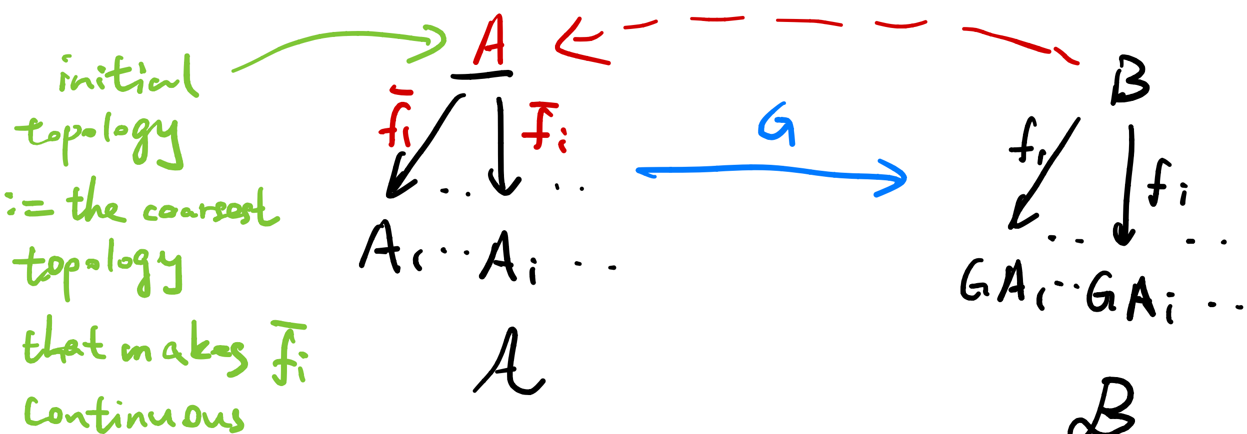
indiscrete $\tau_\nabla = \{\emptyset, X\}$

discrete $\tau_\Delta = \mathcal{P}(X)$

X

Def. (topological functor & topological category):

A functor $A \xrightarrow{G} \mathcal{B}$ is called topological provided that every G-structured source $(B \xrightarrow{f_i} GA_i)_I$ has a unique G-initial lift $(A \xrightarrow{\bar{f}_i} A_i)_I$. A concrete category $(\mathcal{A}, U)$ is called topological provided that U is topological.



a confusion from me on the word 'initial'

$$
\begin{tikzcd}
\overline{A} \arrow[d, "\bar f_1"'] \arrow[dr, "\bar F_i"] & & B \arrow[d, "f_1"'] \arrow[dr, "f_i"] \arrow[ll, dashed, red] & \\
A_1 \cdots & A_i \cdots & GA_1 \cdots & GA_i \cdots
\end{tikzcd}
$$

$\mathcal{A} \xrightarrow{G} \mathcal{B}$

initial topology := the coarsest topology that makes $\bar{f}_i$ continuous

$\longleftarrow$ a confusion from me on the word 'initial'

$\tau_\nabla = \{\emptyset, X\}$

$\vdots$

$\tau_\Delta = \mathcal{P}(X)$

(weak) coarse indiscrete $\to$ $\tau_\nabla = \{\emptyset, X\}$

$\updownarrow$

(strong) fine discrete $\to$ $\tau_\Delta = \mathcal{P}(X)$

$\tau \subseteq \tau'$ iff $\tau \leq \tau'$ instead!

X $\leftarrow$ the underlying set

Topological Duality Theorem

If $(\mathcal{A}, U)$ is topological over $\mathcal{X}$, then $(\mathcal{A}^{op}, U^{op})$ is topological over $\mathcal{X}^{op}$.

the existence of unique U-initial lifts of U-structured sources implies the existence of unique U-final lifts of U-structured sinks.

coproduct as in Set

the finest to make π_1 & π_2 continuous

$$
\begin{tikzcd}
& (\underline{X_1 \sqcup X_2}, \underline{\tau'}) & \\
(X_1, \tau_1) \arrow[ur, "p_1"] & & (X_2, \tau_2) \arrow[ul, "p_2"']
\end{tikzcd}
$$

Some Useful Implications

Sup $\mathcal{A} \xrightarrow{U} \mathcal{B}$

- The U-fibre over any object in $\mathcal{B}$ is a complete lattice. $\rightarrow$ think of  in Top.
- U lifts (co)limits uniquely. $\rightarrow$ think of (co)product topology
- If $\mathcal{B}$ is (co)complete, so is $\mathcal{A}$. $\rightarrow$ Set is (co)complete

  $\Downarrow$

  Top is also

Examples

Top :

the coarsest that makes the inclusion i continuous.

the subspace topology $\quad i : (Y, \mathcal{T}_Y) \hookrightarrow (X, \mathcal{T}_X)$

the finest that makes the surjective map π continuous

the quotient topology $\quad \pi : (X, \mathcal{T}_X) \twoheadrightarrow (S, \mathcal{T}_S)$

the product topology $\quad$ the coarsest ..

the coproduct topology $\quad$ the finest ..

Kr (DiGph)

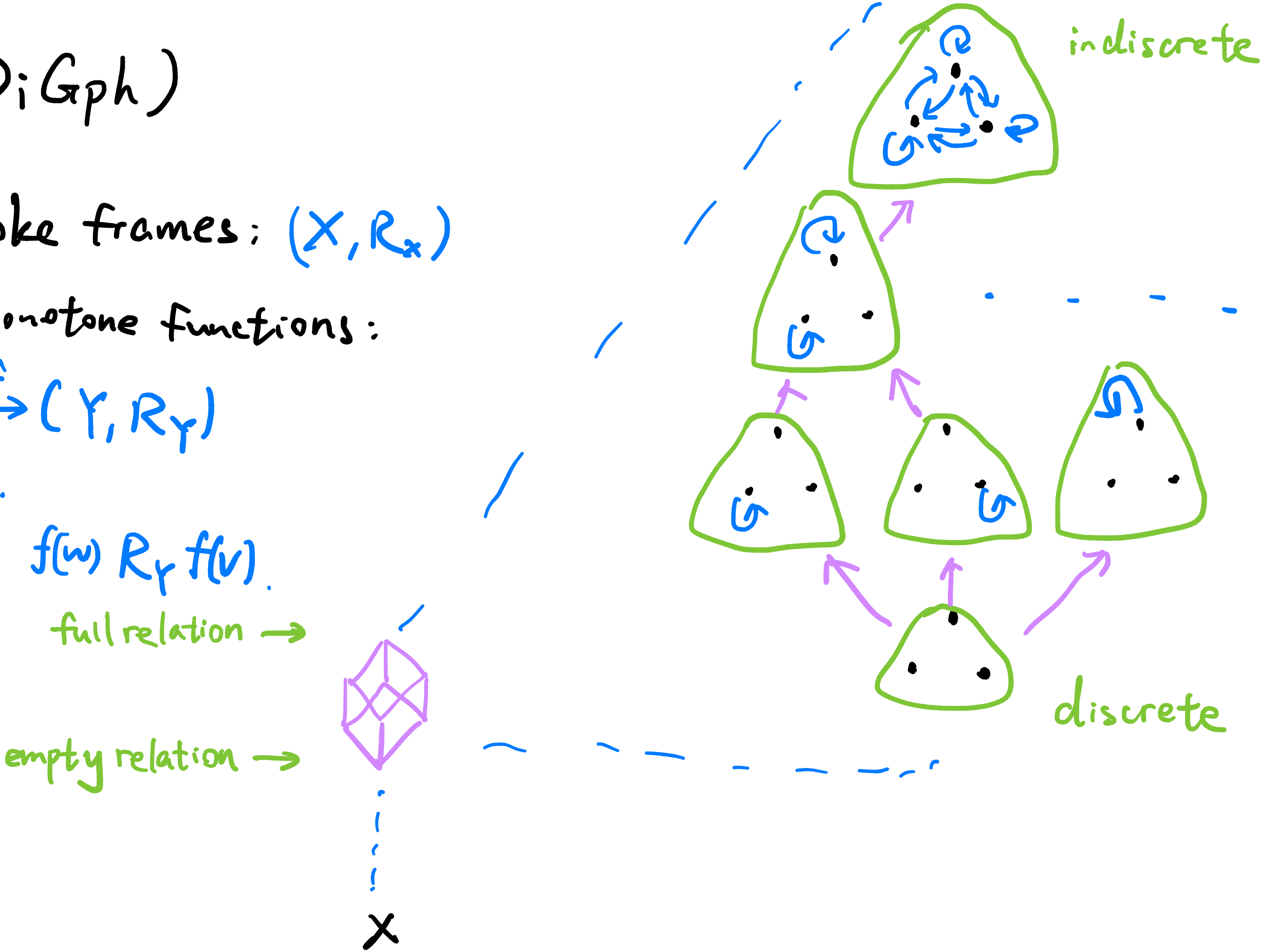
Objects: Kripke frames: (X, R_X)

Morphisms: Monotone functions:

$$(X, R_X) \xrightarrow{f} (Y, R_Y)$$

s.t.

$$w R_X v \Rightarrow f(w) R_Y f(v).$$



More examples & non-examples

The category of irreflexive graphs ✗

It has no terminal object

$\Downarrow$

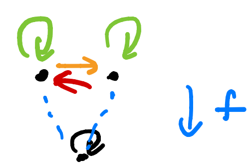
not complete

Initial lifts preserve refl., trans, symmetry...

$\Downarrow$

the corresponding full subcategories of Kr like Preord, Equiv. ✓

But not antisymmetry



$\Downarrow$ (no unique initial lifts)

the category of posets is not topological over Sets ✗

Nb

Objects: Neighbourhood frames: (X, E)

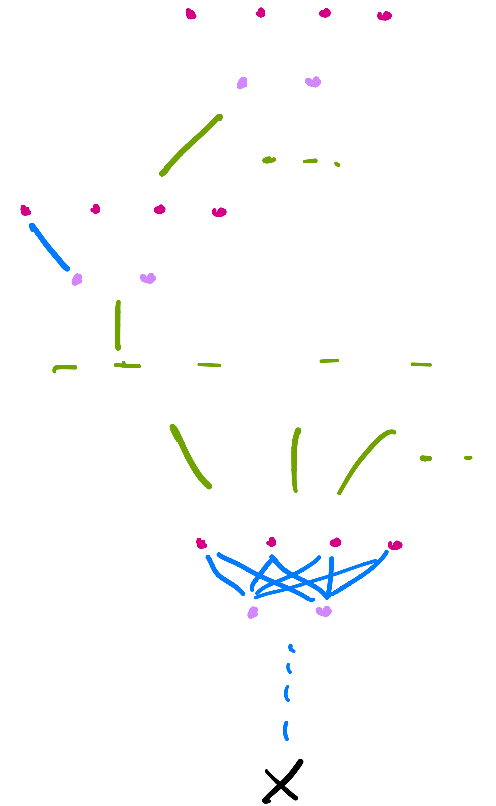
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cong$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad X \times \mathcal{P}(X)$

Morphisms:

$f : (X, E) \to (Y, F)$ satisfying a continuity condition:

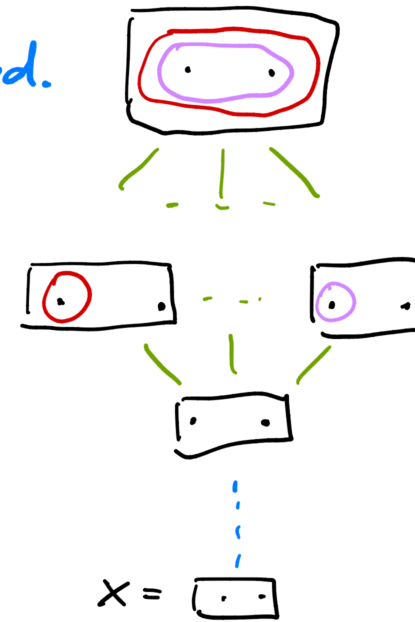
$\forall x \in X,\ V \subseteq Y.\ \ fx\,F\,V \Rightarrow x\,E\,f^{-1}V.$



$\mathcal{P}(X) = \{\emptyset, \{w\}, \{u\}, \{w, u\}\}$

$X = \{w, u\}$

Evl_P ← with a set P of proposition variables fixed.



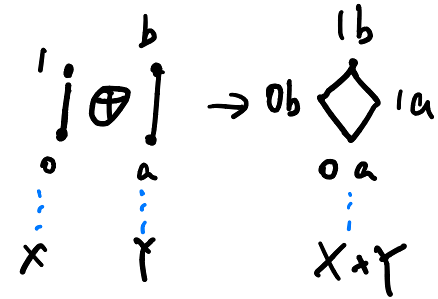
Objects: (X, V)

$V : P \to \mathcal{P}(X)$

Morphisms: $f : (X, V) \to (Y, W)$

satisfying $V \subseteq f^{-1} \circ W$

Fact: The product of a family $\{A_i\}_{i \in I}$ of topological categories can still be seen as a topological category.



e.g. Frame $\longrightarrow$ Model

$F : \mathrm{Kr} \longrightarrow (F, V) : \mathrm{Kr} \times \mathrm{Evl}$

Applications

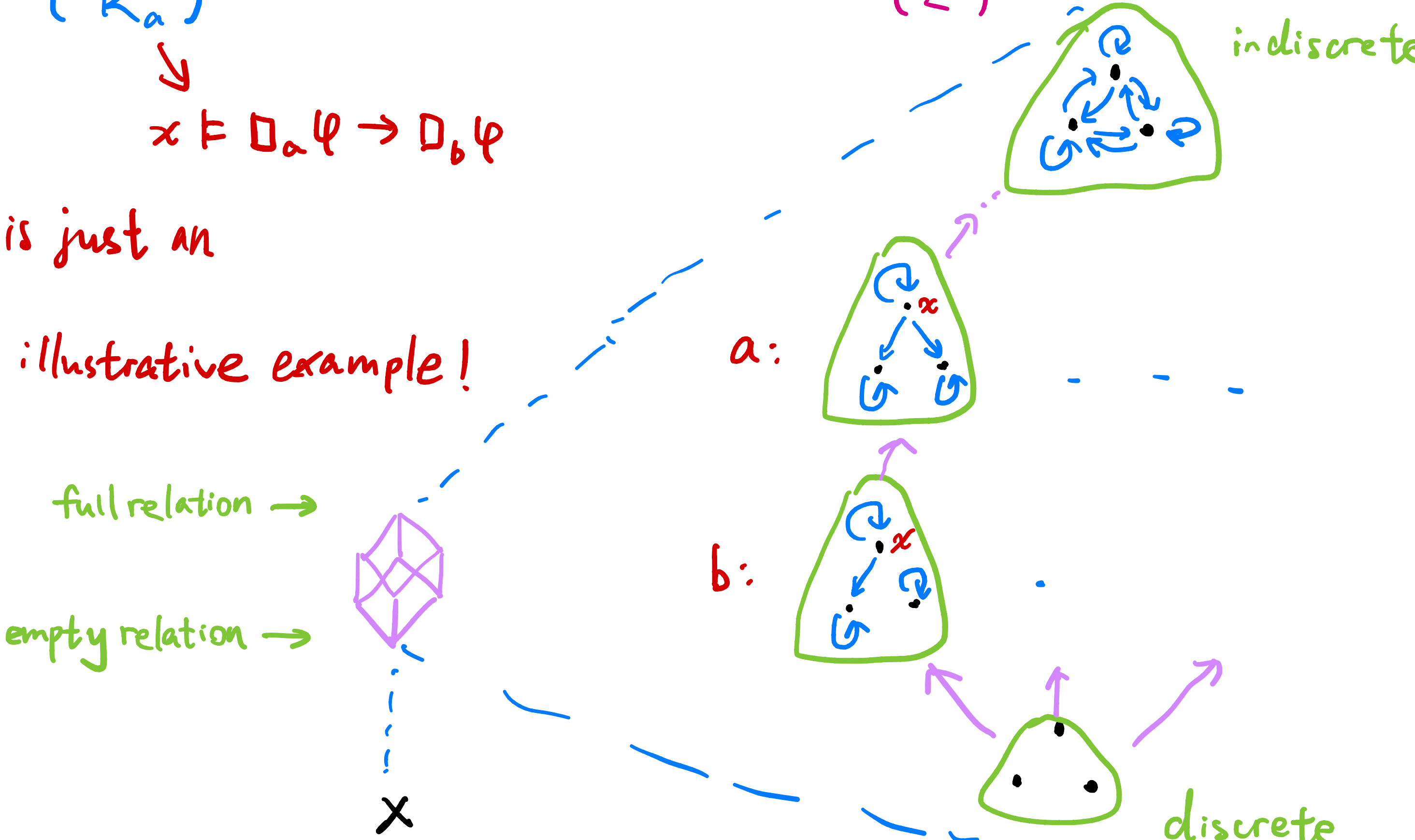
Modal Reasoning Patterns		Universal structures in Topcat
Modal Dependence (K_a^b)	$\leftrightarrow$	Partial Orders within Fibres ($\leq$)
Group Structure (common/distributed knowledge)	$\leftrightarrow$	Lattice operations within Fibres ($\vee$, $\wedge$)
Logical Dynamics (public announcement, product update)	$\leftrightarrow$	Connections between fibres (f^*, $f_!$)

Modal Dependence $\longleftrightarrow$ Partial Orders within Fibres

(K_a^b) $\qquad\qquad\qquad\qquad$ ($\leq$)

$x \models \Box_a \varphi \to \Box_b \varphi$

Kr is just an illustrative example!

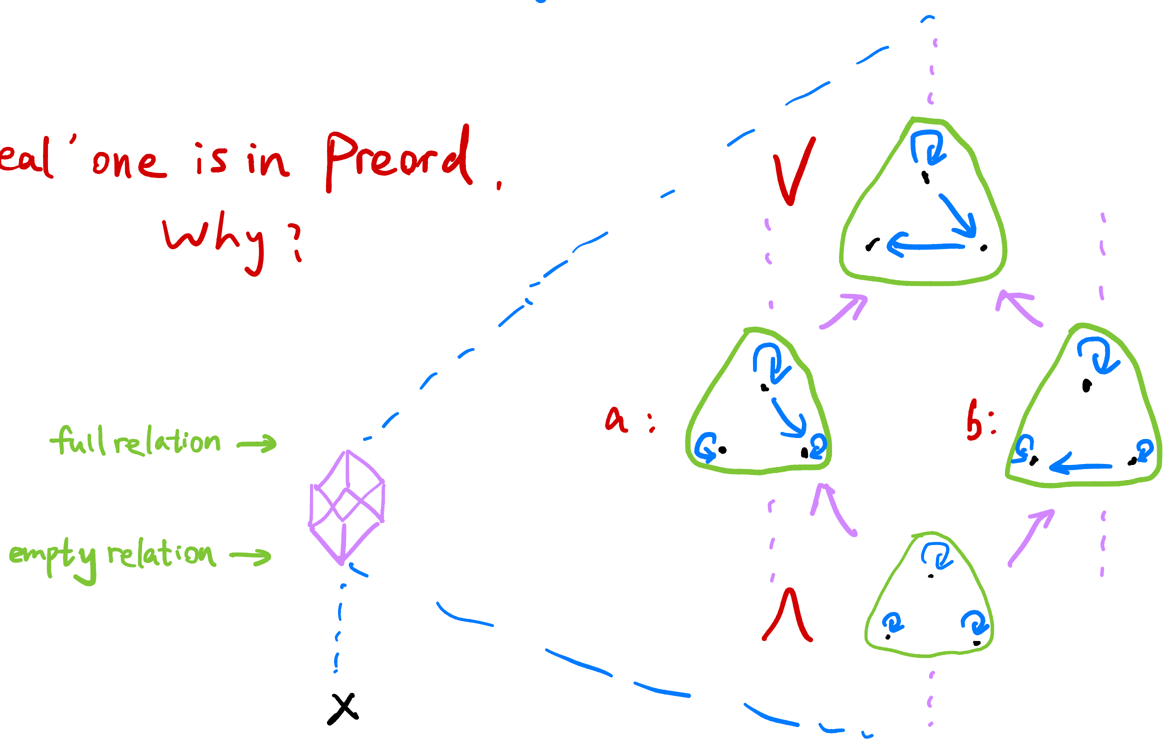


Group Structure $\longleftrightarrow$ Lattice operations within Fibres

(common/distributed knowledge) $(\vee, \wedge)$

the 'real' one is in Preord.

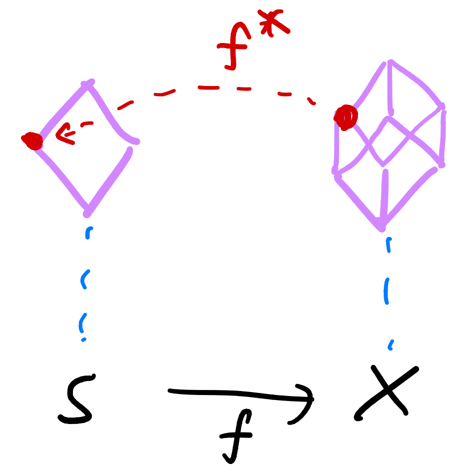
why?



Logical Dynamics $\longleftrightarrow$ Connections between fibres

(Public announcement) $\qquad\qquad$ $(f^*, f_!)$

$$
\begin{tikzcd}
\overbrace{(S, R_S}^{S'}, V_S) \arrow[r, "i"] & (\overbrace{X, R_X}^{X'}, V) \arrow[l, bend left, "i^*"]
\end{tikzcd}
$$



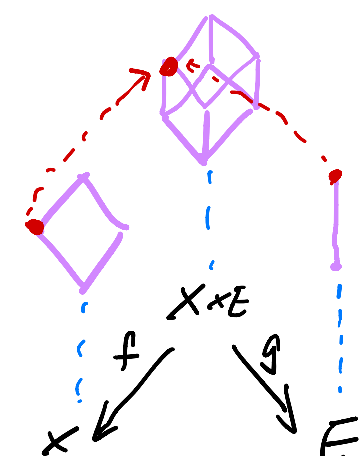
$$X', V, w \models [!\varphi]\psi \iff X', V, w \models \varphi \text{ implies } i^*X', i^*V, w \models \psi$$

Logical Dynamics $\longleftrightarrow$ Connections between fibres

(product update) $\qquad\qquad$ $(f^*, f_!)$

$$
\begin{tikzcd}
X\otimes E = \sum_{e\in E} [\![Pre(e)]\!]_X \arrow[r, hook, "i"] \arrow[rr, bend left, "p_E"] \arrow[dr, "p_X"] & X\times E \arrow[r, "p_E'"] \arrow[d, shift left, "p_X'"] & E \\
{[\![Pre(e)]\!]_X} \arrow[u, "q_e"] \arrow[r, "i_e"'] \arrow[ur, phantom, "\lrcorner", very near start] & X \arrow[u, shift left, "q_{e'}"] \arrow[ul, bend left, "R_e" description] &
\end{tikzcd}
$$

coproduct injections



Modal Reasoning Patterns	Universal structures in TopCat

Modal Dependence $\longleftrightarrow$ Partial Orders within Fibres

(K_a^b) ($\leq$)

Group Structure $\longleftrightarrow$ Lattice operations within Fibres

(common/distributed knowledge) ($\vee$, $\wedge$)

Logical Dynamics $\longleftrightarrow$ Connections between fibres

(public announcement, product update) (f^*, $f_!$)

We can translate between semantic models!

$$\mathcal{A} \xrightarrow{F} \mathcal{B}$$

e.g. $\mathbf{Pre} \hookrightarrow \mathbf{Top}$

sends a preorder to its Alexandroff topology

$[\![\varphi]\!]^{\vee}_{A} = [\![\varphi]\!]^{\vee}_{FA}$

Model transformation F that preserves the interpretation of syntactic pattern $\iff$ F commutes with the semantic structure $\mathcal{A}$ corresponds to.

K^{b}_{a}

$[!\varphi]\psi$

$[E,e]\varphi$

preserves arbitrary meets fibre-wise

preserve the initial lifts of injections

+ preserves pullback maps & fibre-wise finite meets

$$\mathcal{A} \xrightarrow{F} \mathcal{B}$$

$[\![\varphi]\!]^\vee_{\mathcal{A}} = [\![\varphi]\!]^\vee_{F\mathcal{A}}$

Model transformation F that preserves the interpretation of syntactic pattern $\iff$ F commutes with the semantic structure $\mathcal{A}$ corresponds to.

K_a^b

$[!\varphi]\psi$

$[E,e]\varphi$

preserves arbitrary meets fibre-wise

Preserve the initial lifts of injections

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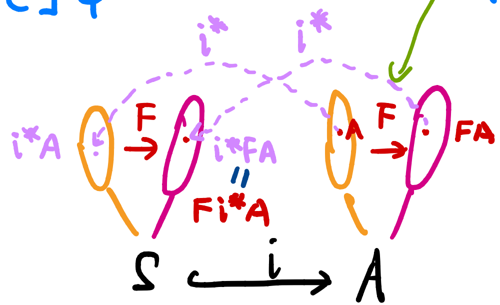


What I didn't cover:

- View topological categories as functors $\mathcal{A}\_ : \mathbf{Set} \to \mathbf{SupLat}$
  via Grothendieck construction or $A\_ : \mathbf{Set}^{op} \to \mathbf{InfLat}$

- CABAO giving the algebraic semantics
  (Kr, Top, Nb can be embedded into CABAO)

- Generalized product update

$$\begin{tikzcd} S \otimes_V X \arrow[r, hook, "i"] \arrow[d, "\pi_S"'] & E \otimes_V X \arrow[d, "\pi_E"] \\ S \arrow[r, hook, "i"] & E \end{tikzcd}$$

$$\Downarrow$$

$$\begin{tikzcd} Y \otimes_V X \arrow[r, "g"] \arrow[d, "\pi_Y"'] & E \otimes_V X \arrow[d, "\pi_E"] \\ Y \arrow[r, "f"] & E \end{tikzcd}$$

- ...

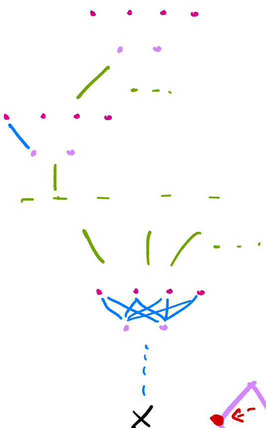
Reference

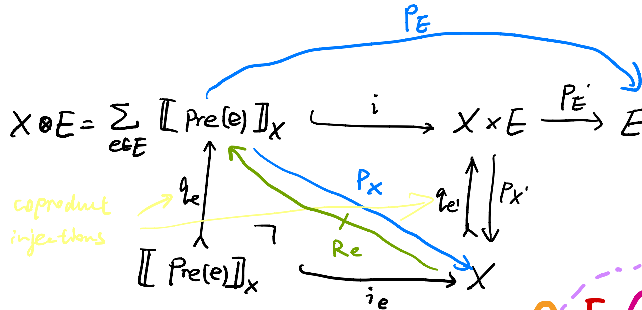
Topology : a categorical approach

The joy of cats

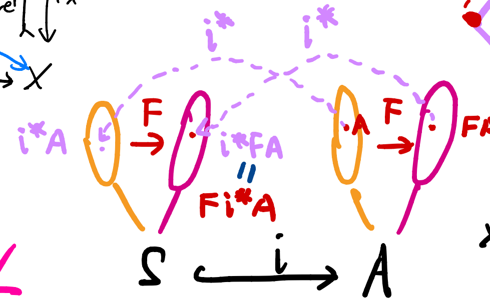
Chris Grossack's blogpost on topological categories

☆ Lingyuan's paper on Unification of Modal Logic

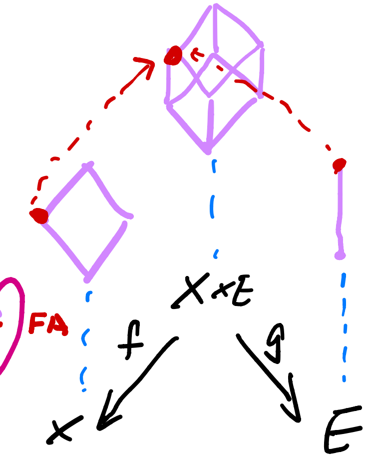




$X \otimes E = \sum_{e \in E} [\![Pre(e)]\!]_X$
i
$X \times E$
p_E'
E
p_E
q_e
$q_{e'}$
p_X
p_X'
R_e
$[\![Pre(e)]\!]_X$
i_e
X
coproduct injections

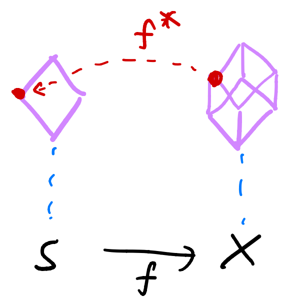


i^*
i^*
i^*A
F
i^*FA
=
Fi^*A
F
A
FA
S
i
A

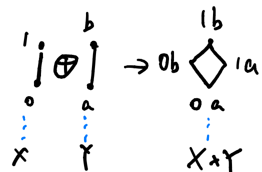


$X \times E$
f
g
X
E

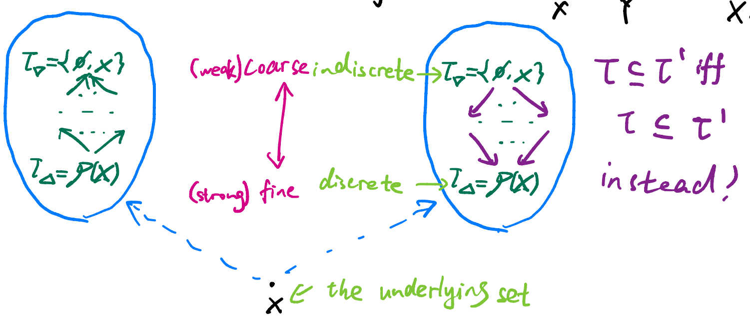
Thanks!



f^*
S
f
X



1
b
1b
0b
1a
0
a
0 a
X
Y
$X \times Y$



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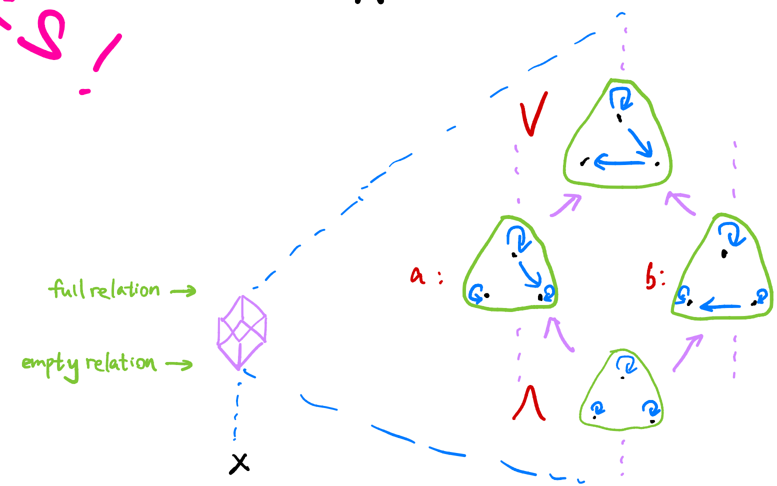
(weak) coarse indiscrete
(strong) fine
discrete
$\tau_\nabla = \{\emptyset, X\}$
$\tau_\Delta = \mathcal{P}(X)$
$\tau \subseteq \tau'$ iff
$\tau \leq \tau'$
instead!
X
the underlying set



full relation
empty relation
X
V
a :
b:
Λ